GENERATION VIA VARIATIONAL CONVERGENCE OF BALANCED VISCOSITY SOLUTIONS TO RATE-INDEPENDENT SYSTEMS

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Abstract. In this paper we investigate the origin of the Balanced Viscosity solution concept for rate-independent evolution in the setting of a finite-dimensional space. Namely, given a family of dissipation potentials $(\Psi_n)_n$ with superlinear growth at infinity and a smooth energy functional $\mathcal{E}$, we enucleate sufficient conditions on them ensuring that the associated gradient systems $(\Psi_n, \mathcal{E})$ *Evolutionary Gamma-converge*, cf. [Mie16], to a limiting rate-independent system, understood in the sense of *Balanced Viscosity* solutions. In particular, our analysis encompasses both the *vanishing-viscosity* approximation of rate-independent systems from [MRS12a, MRS16], and their *stochastic* derivation developed in [BP16].

Key words: Gradient Systems, Rate-Independent Systems, Balanced Viscosity solutions, Vanishing Viscosity, Large Deviations, Variational Convergence.

1. Introduction

Over the last years, rate-independent systems have been the object of intensive mathematical investigations. This is undoubtedly due to their vast range of applicability. Indeed, this kind of processes seems to be ubiquitous in continuum mechanics, ranging from shape memory alloys to crack propagation, from elastoplasticity to damage and delamination. They also occur in fields such as ferromagnetism and ferroelectricity. We refer to [Mie05, MR15] for a thorough survey of all these problems.

Besides its applicative relevance, though, rate-independent evolution has an own, intrinsic, mathematical interest. This is apparent already in the context of a *finite-dimensional* rate-independent system, driven by a *dissipation potential* $\Psi_0 : \mathbb{R}^d \to [0, +\infty)$ (non-degenerate), convex, and positively homogeneous of degree 1, and an *energy functional* $\mathcal{E} : [0, T] \times \mathbb{R}^d \to \mathbb{R}$; in particular, throughout the paper, we will consider a smooth energy $\mathcal{E}$ such that the power function $\partial_t \mathcal{E}$ is controlled by $\mathcal{E}$ itself, namely

$$\mathcal{E} \in \mathrm{C}^1([0,T] \times \mathbb{R}^d) \quad \text{and} \quad \exists\, C_1,\, C_2 > 0 \ \ \forall\, (t,u) \in [0,T] \times \mathbb{R}^d : \qquad |\partial_t \mathcal{E}(t,u)| \leq C_1 \mathcal{E}(t,u) + C_2 \,. \tag{E}$$

The pair $(\Psi_0, \mathcal{E})$ give rise to the simplest example of rate-independent evolution, namely the gradient system

$$\partial \Psi_0(u'(t)) + \mathrm{D}\mathcal{E}(t, u(t)) \ni 0 \qquad \text{for a.a. } t \in (0,T), \tag{1.1}$$

where $\partial \Psi_0 : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ is the subdifferential of Ψ_0 in the sense of convex analysis, whereas $\mathrm{D}\mathcal{E}$ is the differential of the map $u \mapsto \mathcal{E}(t,u)$. Due to the 0-homogeneity of $\partial\Psi_0$, (1.1) is invariant for time rescalings, i.e. it is rate-independent. Now, it is well known that, even in the case of a *smooth* energy $\mathcal{E}$, if $u \mapsto \mathcal{E}(t,u)$ fails to be strictly convex, then absolutely continuous solutions to (1.1) need not exist. In the last two decades, this has motivated the development of various weak solvability concepts for (1.1) and, in general, for rate-independent systems in infinite-dimensional Banach spaces, or even topological spaces. The analysis of these solution notions has posed several challenges.

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***Energetic* and *Balanced Viscosity* solutions to rate-independent systems.** While referring to [Mie11] and [MR15] for a survey of all weak notions of rate-independent evolution, we may recall here the concept of *Energetic solution*, first proposed in [MT99] (cf. also [DMT02] for the concept of *quasistatic evolution* in fracture), and fully analyzed in [MT04]. It consists of the *global stability* condition

$$\forall\, z \in \mathbb{R}^d: \qquad \mathcal{E}(t,u(t)) \leq \mathcal{E}(t,z) + \Psi_0(z-u(t)) \qquad \text{for every } t \in [0,T], \tag{S}$$

and of the $(\Psi_0,\mathcal{E})$-*energy balance*

$$\mathcal{E}(t,u(t)) + \mathrm{Var}_{\Psi_0}(u;[0,t]) = \mathcal{E}(0,u(0)) + \int_0^t \partial_t \mathcal{E}(s,u(s))\,\mathrm{d}s \qquad \text{for every } t \in [0,T], \tag{$\mathrm{E}_{\Psi_0,\mathcal{E}}$}$$

involving the dissipated energy $\mathrm{Var}_{\Psi_0}(u;[0,t])$ (where Var_{Ψ_0} denotes the notion of total variation induced by Ψ_0), the stored energy $\mathcal{E}(t,u(t))$ at the process time t, the initial energy $\mathcal{E}(0,u(0))$, and the work of the external forces. Since the energetic formulation (S)–($\mathrm{E}_{\Psi_0,\mathcal{E}}$) only features the (assumedly smooth) power of the external forces $\partial_t\mathcal{E}$, and no other derivatives, it is particularly suited to solutions with discontinuities in time. It is also considerably flexible and can be indeed given for rate-independent processes in general topological spaces, cf. [MM05]. That is why, it has been exploited in a great variety of applicative contexts, cf. [Mie05, MR15].

Nonetheless, over the years it has become apparent that, in the very case of a *nonconvex* dependence $u \mapsto \mathcal{E}(t,u)$, the *global* stability (S) fails provide a truthful description of the system behaviour at jumps, leading to solutions jumping 'too early' and 'too long' (i.e. into very far-apart energetic configurations), as shown for instance by the examples [Mie03, Ex. 6.1], and [MRS09, Ex. 1], and by the characterization of energetic solutions to (one-dimensional) rate-independent systems in [RS13].

This circumstance has led to the introduction of *alternative* weak solvability concepts for (1.1) and its generalizations. The focus of this paper is on the notion of *Balanced Viscosity* solution, first introduced in [MRS12a] for a *finite-dimensional* rate-independent system and later extended to the infinite-dimensional case in [MRS16]. The origin of this concept in fact goes back to the seminal paper [EM06], which first set forth *vanishing viscosity* as a selection criterion for mechanically feasible weak solution notions to rate-independent systems. The vanishing-viscosity approach has in fact proved to be a robust method in manifold applications, e.g. ranging from plasticity [DDS11, BFM12, FS13], to fracture [KMZ08, LT11], and to damage [KRZ13, CL16] models. We also refer to [Neg14] for an alternative derivation of Balanced Viscosity solutions via time discretization.

Let us briefly illustrate the vanishing-viscosity approach: We "augment by viscosity" the dissipation potential Ψ_0 and thus introduce

$$\Psi_\varepsilon(v) := \Psi_0(v) + \frac{\varepsilon}{2}\|v\|^2, \tag{1.2}$$

with $\|\cdot\|$ a second norm on $\mathbb{R}^d$, possibly coinciding with Ψ_0, and the corresponding gradient system $(\Psi_\varepsilon,\mathcal{E})$, namely the doubly nonlinear equation

$$\partial\Psi_\varepsilon(u'(t)) + \mathrm{D}\mathcal{E}(t,u(t)) \ni 0 \qquad \text{for a.a. } t \in (0,T). \tag{1.3}$$

Since Ψ_ε has superlinear growth at infinity, (1.3) does admit absolutely continuous solutions. It is to be expected that, as the viscosity parameter ε vanishes, solutions $(u_\varepsilon)_\varepsilon$ to (1.3) will converge to a suitable weak solution to the rate-independent system (1.1). In [MRS12a] it was indeed shown that any limit curve $u \in \mathrm{BV}([0,T];\mathbb{R}^d)$ of the functions $(u_\varepsilon)_\varepsilon$ complies with the stability condition

$$-\,\mathrm{D}\mathcal{E}(t,u(t)) \in K^* := \partial\Psi_0(0) \quad \text{for a.a. } t \in (0,T), \tag{$\mathrm{S_{loc}}$}$$

and with the energy balance

$$\mathrm{Var}_{\Psi_0,\mathfrak{p},\mathcal{E}}(u;[0,t]) + \mathcal{E}(t,u(t)) = \mathcal{E}(0,u(0)) + \int_0^t \partial_t\mathcal{E}(s,u(s))\,\mathrm{d}s \quad \text{for all } t \in [0,T]\,. \tag{$\mathrm{E}_{\Psi_0,\mathfrak{p},\mathcal{E}}$}$$

Although ($\mathrm{S_{loc}}$) & ($\mathrm{E}_{\Psi_0,\mathfrak{p},\mathcal{E}}$) look similar to (S) & ($\mathrm{E}_{\Psi_0,\mathcal{E}}$), they are in fact significantly different. First of all, ($\mathrm{S_{loc}}$) is in fact a *local* version of the global stability (S). Secondly, ($\mathrm{E}_{\Psi_0,\mathfrak{p},\mathcal{E}}$) shares the same structure with the energy balance ($\mathrm{E}_{\Psi_0,\mathcal{E}}$), but it features a notion of total variation involving, in addition to the dissipation potential Ψ_0, the *vanishing-viscosity contact potential*

$$\mathfrak{p}(v,\xi) := \inf_{\varepsilon>0} \left(\Psi_\varepsilon(v) + \Psi_\varepsilon^*(\xi)\right) = \Psi_0(v) + \|v\| \min_{\zeta\in K^*} \|\xi-\zeta\|_*, \tag{1.4}$$

(with K^* from ($\mathrm{S_{loc}}$) and $\|\cdot\|_*$ the dual norm of $\|\cdot\|$). While referring to Section 3 for the precise definition of $\mathrm{Var}_{\Psi_0,\mathfrak{p},\mathcal{E}}$, cf. (3.20), we may mention here that $\mathfrak{p}$ indeed encodes how viscosity, neglected in the vanishing-viscosity limit, pops back into the description of the solution behaviour at jumps, whereas in the continuous ('sliding') regime, the system is only governed by the dissipation Ψ_0.

A characterization of Balanced Viscosity solutions, again for one-dimensional systems, has been provided in [RS13], showing that they model jumps more accurately than energetic solutions. On the other hand, as evident from (1.4), this notion seems to be strongly reminiscent of the vanishing-viscosity approximation (1.3).

It is thus natural to wonder if there are ways, *alternative* to the vanishing-viscosity [MRS12a, MRS16] and to the time-discretization [Neg14] approaches, to generate Balanced Viscosity solutions.

The stochastic origin of Balanced Viscosity solutions. Recently, this question has been answered affirmatively in [BP16], investigating the role of stochasticity in the origin of rate-independence, in the *one-dimensional* setting (we refer to [MPR14] for analogous results on the origins of generalized gradient structures). More specifically, [BP16] has focused on a continuous-time Markov jump process $t\mapsto X_t^h$ on a one-dimensional lattice, with lattice spacing $\frac{1}{h}$, $h\in\mathbb{N}$. While referring to Section 2 for more details, we may mention here that this process models the evolution of a Brownian particle in a wiggly energy landscape, involving the energy $\mathcal{E}$, in the following way. If the particle is at the position x at time t, then it jumps in continuous time to its neighbours $x\pm\frac{1}{h}$ with rates $hr^{\pm}(x)$, where $r^{\pm}(x) = \alpha\exp(\mp\beta\mathrm{D}\mathcal{E}(t,x))$. Here, α and β are positive parameters, the former characterizing the rate of jumps, and thus the global time scale of the process, and the latter related to noise.

First of all, in [BP16] it was shown that the deterministic limit, in a 'large-deviations' sense, as $h\to\infty$ and for α and β fixed, of this stochastic process solves the gradient system

$$u'(t) = 2\alpha\sinh(-\beta\mathrm{D}\mathcal{E}(t,u(t))) \qquad \text{for a.a. } t\in(0,T).$$

Observe that the latter is a reformulation of the doubly nonlinear evolution equation

$$\partial\Psi_{\alpha,\beta}(u'(t)) + \mathrm{D}\mathcal{E}(t,u(t)) \ni 0 \qquad \text{for a.a. } t\in(0,T), \tag{1.5}$$

where the dissipation potential $\Psi_{\alpha,\beta}$ is such that its Fenchel-Moreau convex conjugate fulfills $\partial\Psi_{\alpha,\beta}^*(\xi) = \{\mathrm{D}\Psi_{\alpha,\beta}^*(\xi)\} = \{2\alpha\sinh(\beta\xi)\}$. More precisely, in [BP16] it was proved that the process X^h satisfies a *large deviations principle*, with rate function given by the functional of trajectories $\tilde{\mathscr{J}}_{\Psi_{\alpha,\beta},\mathcal{E}} : \mathrm{BV}([0,T];\mathbb{R}^d) \to [0,+\infty]$ defined by

$$\tilde{\mathscr{J}}_{\Psi_{\alpha,\beta},\mathcal{E}}(u) := \beta\left(\int_0^T \left(\Psi_{\alpha,\beta}(u'(t)) + \Psi_{\alpha,\beta}^*(-\mathrm{D}\mathcal{E}(t,u(t)))\right)\mathrm{d}t + \mathcal{E}(T,u(T)) - \mathcal{E}(0,u(0)) - \int_0^T \partial_t\mathcal{E}(t,u(t))\,\mathrm{d}t\right)$$

if $u\in\mathrm{AC}([0,T];\mathbb{R}^d)$, and $+\infty$ else. It is easy to check that the (null-)minimizers of $\tilde{\mathscr{J}}_{\Psi_{\alpha,\beta},\mathcal{E}}$ are solutions to the gradient system (1.5).

Next, the *variational* limits of the functionals $\tilde{\mathscr{J}}_{\Psi_{\alpha,\beta},\mathcal{E}}$ have been addressed under different scalings of the parameters α and β, leading to gradient flow or rate-independent evolution. To illustrate the result in the latter case, here and throughout the paper we will confine the discussion to the following choice of parameters: $\alpha = \alpha_n := \frac{e^{-nA}}{2}$ and $\beta := \beta_n = n$, with $n\in\mathbb{N}$. Therefore, the associated dissipation potentials are given by

$$\Psi_n(v) := \Psi_{\alpha_n,\beta_n}(v) = \frac{v}{n}\log\left(\frac{v+\sqrt{v^2+e^{-2nA}}}{e^{-nA}}\right) - \frac{1}{n}\sqrt{v^2+e^{-2nA}} + \frac{e^{-nA}}{n}. \tag{1.6}$$

In [BP16, Thm. 4.2] it was then proved that that the functionals $\mathscr{J}_{\Psi_n,\mathcal{E}} := \frac{1}{n}\tilde{\mathscr{J}}_{\Psi_n,\mathcal{E}}$ converge in the sense of Mosco, with respect to the *weak-strict* topology in $\mathrm{BV}([0,T];\mathbb{R}^d)$, to the functional $\mathscr{J}_{\Psi_0,\mathfrak{p},\mathcal{E}} : \mathrm{BV}([0,T];\mathbb{R}^d) \to$

$[0,+\infty]$ defined by

$$
\begin{aligned}
\mathscr{J}_{\Psi_0,\mathfrak{p},\mathcal{E}}(u) :=&\mathrm{Var}_{\Psi_0,\mathfrak{p},\mathcal{E}}(u;[0,T]) + \int_0^T I_{K^*}(-\mathrm{D}\mathcal{E}(t,u(t)))\,\mathrm{d}t \\
&+ \mathcal{E}(T,u(T)) - \mathcal{E}(0,u(0)) - \int_0^T \partial_t\mathcal{E}(s,u(s))\,\mathrm{d}s,
\end{aligned}
\tag{1.7}
$$

with $\Psi_0(v) = A|v|$, $\mathfrak{p}$ given by (1.4) and the associated total variation functional $\mathrm{Var}_{\Psi_0,\mathfrak{p},\mathcal{E}}$ defined in (3.20) ahead, and with I_{K^*} denoting the indicator function of the set $K^* = [-A,A]$. Recall that MOSCO-convergence (cf. e.g. [Att84]) with respect to the weak-strict topology in $\mathrm{BV}([0,T];\mathbb{R}^d)$ means that

$$
\begin{aligned}
&\textit{(i)} \quad u_n \to u \text{ weakly in } \mathrm{BV}([0,T];\mathbb{R}^d) \ \Rightarrow\ \liminf_{n\to\infty} \mathscr{J}_{\Psi_n,\mathcal{E}}(u_n) \geq \mathscr{J}_{\Psi_0,\mathfrak{p},\mathcal{E}}(u),\\
&\textit{(ii)} \quad \forall\, u \in \mathrm{BV}([0,T];\mathbb{R}^d)\ \exists\,(u_n)_n \subset \mathrm{BV}([0,T];\mathbb{R}^d) \text{ s.t. } \begin{cases} u_n \to u \text{ strictly in } \mathrm{BV}([0,T];\mathbb{R}^d)\,, \\ \limsup_{n\to\infty} \mathscr{J}_{\Psi_n,\mathcal{E}}(u_n) \leq \mathscr{J}_{\Psi_0,\mathfrak{p},\mathcal{E}}(u)\,. \end{cases}
\end{aligned}
\tag{1.8}
$$

Since the (null-)minimizers of $\mathscr{J}_{\Psi_0,\mathfrak{p},\mathcal{E}}$ are Balanced Viscosity solutions of the rate-independent system driven by Ψ_0 and $\mathcal{E}$ (cf. Proposition 3.6 ahead), [BP16, Thm. 4.2] ultimately establishes a connection between the jump process X^h and the latter rate-independent system, understood in a *Balanced Viscosity* sense. Furthermore, observe that the functionals Ψ_n from (1.6) are not of the form (1.2). Therefore, this result provides a way, alternative to vanishing viscosity, to generate Balanced Viscosity solutions.

Our results. The aim of this paper is twofold.

First of all, we intend to extend the 'stochastic generation' of Balanced Viscosity solutions investigated in [BP16], to the *multi-dimensional* rate-independent system (1.1), where now

$$
\Psi_0(v) := A\|v\|_1 \text{ for all } v \in \mathbb{R}^d, \qquad \text{with } \|v\|_1 := \sum_{i=1}^d |v_i|\,.
$$

Even conjecturing that the viscosity contact potential defining the limiting Balanced Viscosity solution notion is of the form (1.4), in the multi-dimensional case it is no longer obvious which choice of the viscous norm $\|\cdot\|$ should enter into (1.4). Indeed, with **our main results, <u>Theorem 5.2</u>** ($\liminf$-estimate) and **<u>Thm. 5.8</u>** ($\limsup$-estimate), we will show that the multi-dimensional analogues of the functionals $(\mathscr{J}_{\Psi_n,\mathcal{E}})_n$ MOSCO-converge, with respect to the weak-strict topology of $\mathrm{BV}([0,T];\mathbb{R}^d)$, to the functional $\mathscr{J}_{\Psi_0,\mathfrak{p},\mathcal{E}}$ featuring the contact potential

$$
\mathfrak{p}(v,\xi) := \|v\|_1(A \vee \|\xi\|_\infty) \qquad \text{with } \|\xi\|_\infty := \max_{i=1,\ldots,d} |\xi_i|. \tag{1.9}
$$

It can be checked that $\mathfrak{p}$ is indeed of the form (1.4), with the 'viscous' norm $\|\cdot\|$ in fact coinciding with that associated with Ψ_0, i.e. $\|v\| = \Psi_0(v) = A\|v\|_1$. Namely, the notion of Balanced Viscosity solution arising from the stochastic approximation coincides with the one obtained by vanishing Ψ_0-*viscosity*, cf. Example 5.4 ahead.

Secondly, we shall investigate on a more general and deeper level the origin of rate-independent evolution in a Balanced Viscosity sense. More precisely,

- we will introduce an 'extended' notion of Balanced Viscosity solution, induced by a general *viscosity contact potential* $\mathsf{p} : [0,+\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0+\infty]$, $\mathsf{p} = \mathsf{p}(\tau,v,\xi)$, cf. Def. 3.1 ahead, such that the contact potentials $\mathfrak{p}(v,\xi)$ from (1.4) are obtained for $\tau = 0$, i.e. $\mathsf{p}(0,v,\xi) = \mathfrak{p}(v,\xi)$ (i.e., $\mathfrak{p}$ is augmented of the time variable);
- we will enucleate a series of conditions under which a sequence $(\Psi_n)_n$ of *general* dissipation potentials with superlinear growth at infinity, not necessarily of the form (1.2) (vanishing-viscosity) or (1.6) (stochastic approximation), give rise to a viscosity contact potential. Such conditions will amount to requiring that the bipotentials $b_{\Psi_n} : [0,+\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0+\infty]$, associated with the functionals Ψ_n, and defined for $\tau > 0$ by $b_{\Psi_n}(\tau,v,\xi) := \tau\Psi_n(\frac{v}{\tau}) + \tau\Psi_n^*(\xi)$ (cf. (4.1) ahead), converge in a suitable *variational sense* to p.

- It will turn out (cf. Theorem 5.2), that under this condition, joint with a suitable uniform coercivity requirement for the functionals $(\Psi_n)_n$, the Γ-$\liminf$ estimate in (1.8) for the associated trajectory functionals $(\mathscr{J}_{\Psi_n,\mathcal{E}})_n$ holds.
- As we will see, this implies that limit curves $u \in \mathrm{BV}([0,T];\mathbb{R}^d)$ of sequences of solutions $(u_n)_n$ to the gradient systems $(\Psi_n, \mathcal{E})$ are Balanced Viscosity solutions to the rate-independent system $(\Psi_0, \mathsf{p}, \mathcal{E})$, i.e. systems $(\Psi_n, \mathcal{E})_n$ Evolutionary Γ-converge, in the sense of [Mie16], to $(\Psi_0, \mathsf{p}, \mathcal{E})$.

Let us clarify that, the $\liminf$-estimate in Theorem 5.2 will be valid in the general setting specified in the above lines, and could be in fact trivially proved for systems set in *any* abstract finite-dimensional Banach space $\mathbf{X}$. Instead, in Theorem 5.8 we will be able to prove the Γ-$\limsup$ inequality only in the *specific* cases of the vanishing-viscosity and the stochastic approximation.

We believe that Theorem 5.8 could be extended to a broader class of dissipation potentials Ψ_n with superlinear growth at infinity, like in the one-dimensional case (cf. [BP16, Thm. 4.2]). However, the proof of the $\limsup$-estimate in the *fully general* case, i.e. under the sole condition that the bipotentials b_{Ψ_n} variationally converge to p, remains an open problem.

Plan of the paper. In Section 2 we discuss the multi-dimensional analogue of the stochastic model considered in [BP16] and (formally) derive the associated dissipation potential Ψ_n and the induced trajectory functional $\mathscr{J}_{\Psi_n,\mathcal{E}}$. Section 3 is devoted to some recaps on BV functions, which are preliminary to the introduction of the *extended* notion of Balanced Viscosity solution to a rate-independent system $(\Psi_0, \mathsf{p}, \mathcal{E})$ (cf. Definition 3.4), with p a viscosity contact potential in the sense of Definition 3.1. We conclude this section by enucleating some basic properties of Balanced Viscosity solutions. In Section 4 we address the generation of a viscosity contact potential starting from a family $(\Psi_n)_n$ of dissipation potentials with superlinear growth at infinity. Our main results, Theorems 5.2 and 5.8, are stated in Section 5 and proved throughout Section 6.

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2. **The stochastic origin of rate-independent systems**

In this section we briefly describe the multi-dimensional extension of the one-dimensional stochastic model for rate-independent evolution considered in [BP16].

We consider a jump process $t \mapsto X^h(t)$ on a d-dimensional lattice, with lattice spacing $\frac{1}{h}$. The evolution of the process can be described as follows: Fix the origin as initial point. If the process is at the position x at time t, then it jumps in continuous time to its neighbours $(x \pm \frac{1}{m}\mathbf{e}_i)$ with rate $mr_i^{\pm}$, for $i = 1, \ldots, d$, where $(\mathbf{e}_1, \ldots, \mathbf{e}_d)$ is the basis of $\mathbb{R}^d$, cf. Figure 1. The jump rates depend on two parameters α and β, and on the partial derivatives $\mathrm{D}_i\mathcal{E} := \mathrm{D}_{x_i}\mathcal{E}$ of a smooth energy functional $\mathcal{E} : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, namely

$$r_i^+(x,t) = \alpha e^{-\beta \mathrm{D}_i\mathcal{E}(x,t))}, \quad r_i^-(x,t) = \alpha e^{\beta \mathrm{D}_i\mathcal{E}(x,t))} \qquad \text{for } i = 1, \ldots, d. \tag{2.1}$$

The choice of the stochastic process (and thus of the jump rates $r_i^{\pm}$) reflects Kramers' formula [Kra40, Ber13, BP16]. Given a particle evolving in a wiggly energy landscape with noise, this formula provides an estimate of the rate of jumps from one energy well to the next one.

We are interested in the continuum limit as $h \to \infty$. With this aim, we apply the method developed by Feng & Kurtz, cf. [FK06], to prove large-deviations principles for Markov processes.

As in [BP16, Sec. 2.5], we will provisionally assume that the jump rates $r^{\pm}$ are constant in space and time, and thus derive the expression of the rate function, and then formally substitute (2.1) into it. Following [FK06], we consider the generator

$$\Omega_h f(x) := \sum_{i=1}^{d} \left[hr_i^+ \left(f(x + \frac{1}{h}\mathbf{e}_i) - f(x) \right) + hr_i^- \left(f(x - \frac{1}{h}\mathbf{e}_i) - f(x) \right) \right]$$

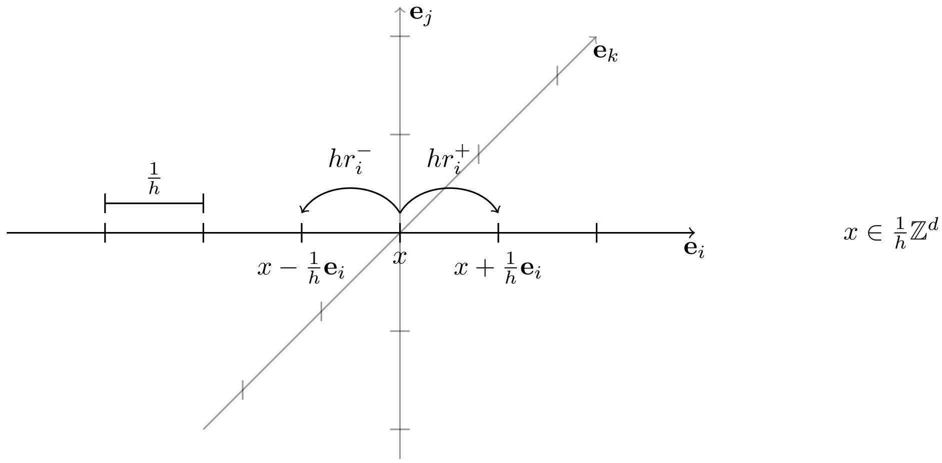

FIGURE 1. A sketch of the jump-process on the lattice.

of the continuous time Markov process X^h, and the nonlinear generator

$$\begin{aligned}(\mathrm{H}_h f)(x) &:= \frac{1}{h} e^{-hf(x)} (\Omega_h e^{hf})(x) \\ &= \sum_{i=1}^{d} \left[r_i^+ \left(\exp\left(h \left(f(x + \frac{1}{h}\mathbf{e}_i) - f(x) \right) \right) - 1 \right) + r_i^- \left(\exp\left(h \left(f(x - \frac{1}{h}\mathbf{e}_i) - f(x) \right) \right) - 1 \right) \right].\end{aligned}$$

According to the Feng-Kurtz method, if H_h converges to some H in a suitable sense, and if the limiting operator $\mathrm{H}f$ depends locally on $\mathrm{D}f$, we can then define the *Hamiltonian* $H = H(x, \xi)$ through

$$(\mathrm{H}f)(x) =: H(x, \mathrm{D}f(x)),$$

and the *Lagrangian* as the Legendre transform of H, namely

$$L(x, v) := \sup_{\xi \in \mathbb{R}^d} \left(\langle \xi, v \rangle - H(x, \xi) \right).$$

Then, the Markov process satisfies a large-deviations principle, with rate function

$$\mathscr{J}(u) := \begin{cases} \int_0^T L(u(t), u'(t))\, \mathrm{d}t & \text{if } u \in \mathrm{AC}([0, T]; \mathbb{R}^d), \\ +\infty & \text{otherwise}, \end{cases} \tag{2.2}$$

cf. [BP16, Sec. 2].

In the present case, it can be seen that

$$H(x, \xi) = \sum_{i=1}^{d} r_i^+ (e^{\xi_i} - 1) + r_i^- (e^{-\xi_i} - 1).$$

Then L is given by

$$L(x, v) = \sum_{i=1}^{d} \left[v_i \log \left(\frac{v_i + \sqrt{v_i^2 + 4 r_i^+ r_i^-}}{2 r_i^+} \right) - \sqrt{v_i^2 + 4 r_i^+ r_i^-} + r_i^+ + r_i^- \right]. \tag{2.3}$$

Substituting in (2.3) the expression (2.1) for the jump rates, and choosing the parameters

$$\alpha = \frac{e^{-nA}}{2} \quad \text{and} \quad \beta = n, \qquad n \in \mathbb{N},$$

we obtain

$$L(x, v) = n \left(\Psi_n(v) + \Psi_n^*(-\mathrm{D}\mathcal{E}(t, x)) + v \mathrm{D}\mathcal{E}(t, x) \right), \tag{2.4}$$

with $\Psi_n : \mathbb{R}^d \to [0, +\infty)$ given by

$$\Psi_n(v) = \sum_{i=1}^{d} \psi_n(v_i) = \sum_{i=1}^{d} \frac{v_i}{n} \log\left(\frac{v_i + \sqrt{v_i^2 + e^{-2nA}}}{e^{-nA}}\right) - \frac{1}{n}\sqrt{v_i^2 + e^{-2nA}} + \frac{e^{-nA}}{n}, \tag{2.5}$$

and Ψ_n^* the Legendre transform of Ψ_n. It can be easily checked that the structure $\Psi_n(v) = \sum_{i=1}^{d} \psi_n(v_i)$ transfers to the conjugate, hence

$$\Psi_n^*(\xi) = \sum_{i=1}^{d} \psi_n^*(\xi_i) = \sum_{i=1}^{d} \frac{e^{-nA}}{n} \left(\cosh(n\xi_i) - 1\right). \tag{2.6}$$

Observe that, with the choice (2.4) for L, the (positive) functional $\mathscr{J}$ from (2.2) is minimized by the solutions of the ODE system (the subdifferential operator $\partial\psi_n^* : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ being single-valued)

$$u_i'(t) = -\mathrm{D}\psi_n^*(-\mathrm{D}_i\mathcal{E}(t, u_i(t))) \qquad \text{for a.a. } t \in (0,T), \text{ for all } i = 1, \cdots, d.$$

3. Viscosity contact potentials and Balanced Viscosity solutions to rate-independent systems

This section is devoted to the notion of *Balanced Viscosity* solution to a rate-independent system. Before introducing it and fixing its main properties, we recall some basic definitions from the theory of functions of bounded variation, and then focus on the crucial concept of *viscosity contact potential*, which underlies the very definition of *Balanced Viscosity* solution.

3.1. Preliminary definitions.

Hereafter, we will call *dissipation potential* any function

$$\Psi : \mathbb{R}^d \to [0, +\infty) \text{ convex and such that } \Psi(0) = 0. \tag{3.1}$$

It follows from the above conditions that the Fenchel-Moreau conjugate Ψ^* then fulfills $\Psi^*(0) = 0 \le \Psi^*(\xi)$ for all $\xi \in \mathbb{R}^d$. We will distinguish two cases:

Dissipation potentials with superlinear growth at infinity i.e. fulfilling

$$\lim_{\|v\| \to +\infty} \frac{\Psi(v)}{\|v\|} = +\infty \tag{3.2}$$

for some norm $\|\cdot\|$ on $\mathbb{R}^d$.

1-homogeneous dissipation potentials In what follows, we will denote by Ψ_0 a dissipation potential

$$\Psi_0 : \mathbb{R}^d \to [0, +\infty) \text{ convex, 1-positively homogenous, and non-degenerate, viz. } \Psi_0(v) > 0 \text{ if } v \neq 0. \tag{3.3}$$

Thus, for any norm $\|\cdot\|$ on $\mathbb{R}^d$

$$\exists\, \eta > 0 \;\forall\, v \in \mathbb{R}^d : \; \eta^{-1}\|v\| \le \Psi_0(v) \le \eta\|v\|. \tag{3.4}$$

Its convex-analysis subdifferential $\partial\Psi_0 : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ at $v \in \mathbb{R}^d$ can be characterized by

$$\zeta \in \partial\Psi_0(v) \;\Leftrightarrow\; \begin{cases} \langle \zeta, w\rangle \le \Psi_0(w) \text{ for all } w \in \mathbb{R}^d, \\ \langle \zeta, v\rangle = \Psi_0(v). \end{cases} \tag{3.5}$$

Throughout, we will use the notation

$$K^* := \partial\Psi_0(0). \tag{3.6}$$

Recall that $\partial\Psi_0(v) \subset K^*$ for all $v \in \mathbb{R}^d$ and that, indeed, Ψ_0 is the *support function* of K^*, namely

$$\Psi_0(v) = \sup_{\zeta \in K^*} \langle \zeta, v\rangle, \quad \text{whence} \quad \Psi_0^*(\xi) = I_{K^*}(\xi). \tag{3.7}$$

BV **functions.** Throughout, we will work with functions of bounded variation *pointwise defined at every point* $t \in [0,T]$. We recall that a function u in $\mathrm{BV}([0,T];\mathbb{R}^d)$ admits left and right limits at every $t \in [0,T]$:

$$u(t_-) := \lim_{s\uparrow t} u(s), \quad u(t_+) := \lim_{s\downarrow t} u(s), \quad \text{with the convention } u(0_-) := u(0), \ u(T_+) := u(T), \tag{3.8}$$

and its *pointwise* jump set J_u is the at most countable set defined by

$$\mathrm{J}_u := \big\{t \in [0,T] : u(t_-) \neq u(t) \text{ or } u(t) \neq u(t_+)\big\} \supset \text{ess-}\mathrm{J}_u := \big\{t \in [0,T] : u(t_-) \neq u(t_+)\big\}. \tag{3.9}$$

We also recall that the distributional derivative u' of u is a Radon vector measure that can be decomposed (cf. [AFP00]) into the sum of the three mutually singular measures

$$u' = u'_{\mathscr{L}} + u'_{\mathrm{C}} + u'_{\mathrm{J}}, \quad u'_{\mathscr{L}} = \dot{u}\,\mathscr{L}^1, \quad u'_{\mathrm{co}} := u'_{\mathscr{L}} + u'_{\mathrm{C}}\,. \tag{3.10}$$

Here, $u'_{\mathscr{L}}$ is the absolutely continuous part with respect to the Lebesgue measure $\mathscr{L}^1$, whose Lebesgue density $\dot{u}$ is the pointwise (and $\mathscr{L}^1$-a.e. defined) derivative of u, u'_{J} is a discrete measure concentrated on ess-$\mathrm{J}_u \subset \mathrm{J}_u$, and u'_{C} is the so-called Cantor part. We will use the notation $u'_{\mathrm{co}} := u'_{\mathscr{L}} + u'_{\mathrm{C}}$ for the diffuse part of the measure, which does not charge J_u.

Given a (non-degenerate) 1-homogeneous dissipation potential Ψ_0, it induces a notion of (pointwise) total variation for a curve $u \in \mathrm{BV}([0,T];\mathbb{R}^d)$ via

$$\mathrm{Var}_{\Psi_0}(u;[a,b]) := \sup\Big\{\sum_{m=1}^{M} \Psi_0\big(u(t_m) - u(t_{m-1})\big) : a = t_0 < t_1 < \cdots < t_{M-1} < t_M = b\Big\} \tag{3.11}$$

for any $[a,b] \subset [0,T]$. Therefore, with any $u \in \mathrm{BV}([0,T];\mathbb{R}^d)$ we can associate the non-decreasing function $V_{\Psi_0} : \mathbb{R} \to [0,+\infty)$ defined by

$$V_{\Psi_0}(t) := \begin{cases} 0 & \text{if } t \leq 0, \\ \mathrm{Var}_{\Psi_0}(u;[0,t]) & \text{if } t \in (0,T), \\ \mathrm{Var}_{\Psi_0}(u;[0,t]) & \text{if } t \geq T. \end{cases}$$

Its distributional derivative μ_{Ψ_0} is in turn a Radon measure that can be decomposed into a jump part $\mu_{\Psi_0,\mathrm{J}}$, concentrated on J_u and given by

$$\mu_{\Psi_0,\mathrm{J}}(\{t\}) = \Psi_0(u(t) - u(t_-)) + \Psi_0(u(t_+) - u(t)),$$

and a diffuse part

$$\mu_{\Psi_0,\mathrm{co}} = \mu_{\Psi_0,\mathscr{L}} + \mu_{\Psi_0,\mathrm{C}} \qquad \text{with} \quad \mu_{\Psi_0,\mathscr{L}} = \Psi_0(\dot{u})\mathscr{L}^1. \tag{3.12}$$

There holds

$$\mathrm{Var}_{\Psi_0}(u;[a,b]) = \mu_{\Psi_0,\mathrm{co}}([a,b]) + \mathrm{Jmp}_{\Psi_0}(u;[a,b]), \tag{3.13}$$

with the jump contribution $\mathrm{Jmp}_{\Psi_0}(u;[a,b])$ given by

$$\begin{aligned} \mathrm{Jmp}_{\Psi_0}(u;[a,b]) &:= \Psi_0(u(a_+) - u(a)) + \mu_{\Psi_0,\mathrm{J}}((a,b)) + \Psi_0(u(b_+) - u(b)) \\ &= \Psi_0(u(a_+) - u(a)) + \sum_{t \in \mathrm{J}_u \cap (a,b)} \Big(\Psi_0(u(t) - u(t_-)) + \Psi_0(u(t_+) - u(t))\Big) + \Psi_0(u(b_+) - u(b))\,. \end{aligned} \tag{3.14}$$

Finally, for later use we recall that a sequence $(u_n)_n$ *weakly* converges in $\mathrm{BV}([0,T];\mathbb{R}^d)$ to a curve u (we will write $u_n \rightharpoonup u$) if $u_n(t) \to u(t)$ as $n \to \infty$ for every $t \in [0,T]$ and $\sup_n \mathrm{Var}(u_n;[0,T]) \leq C < \infty$ (in what follows, we shall denote by $\mathrm{Var}(u;[0,T])$ the total variation of a curve u induced by a generic norm on $\mathbb{R}^d$), whereas $(u_n)_n$ *strictly* converges in $\mathrm{BV}([0,T];\mathbb{R}^d)$ to u ($u_n \to u$) if $u_n \rightharpoonup u$ and $\mathrm{Var}(u_n;[0,T]) \to \mathrm{Var}(u;[0,T])$.

Viscosity contact potentials. The notion we are going to introduce now lies at the core of the definition of *Balanced Viscosity* solution to a rate-independent system, driven by an energy functional $\mathcal{E}$ complying with (E). Indeed, the concept of *viscosity contact potential* encodes how viscosity enters into the description of the solution behavior at jumps. It is an extension of the notion of *vanishing-viscosity contact potential* introduced in [MRS12a], in that we are augmenting the contact potential defined therein by the time variable. In referring to this notion, we will drop the word 'vanishing' in order to highlight that Balanced Viscosity solutions do not necessarily arise from a vanishing-viscosity approximation, cf. Sec. 5.2.

Definition 3.1. *We call a lower semicontinuous function* $\mathsf{p} : [0,+\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0,+\infty]$ (viscosity) contact potential *if it satisfies the following properties:*

(1) for every $\tau \geq 0$ there holds $\mathsf{p}(\tau, v, \xi) \geq \langle v, \xi \rangle$ for all $(v, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$;
(2) for every $\xi \in \mathbb{R}^d$ the map $(\tau, v) \mapsto \mathsf{p}(\tau, v, \xi)$ is convex and positively 1-homogeneous.
(3) for every $\tau > 0$ and $v \in \mathbb{R}^d$, the map $\xi \mapsto \mathsf{p}(\tau, v, \xi)$ is convex.

Moreover, we say that p *is* non-degenerate *if*

(4) for every $\tau \geq 0$ there holds $\mathsf{p}(\tau, v, \xi) > 0$ if $v \neq 0$.

Finally, given a (non-degenerate) 1-homogeneous dissipation potential Ψ_0 as in (3.3), we say that p is Ψ_0-non degenerate *if*

(5) for all $(v, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$ there holds $\mathsf{p}(0, v, \xi) \geq \Psi_0(v)$.

A crucial object related to a (viscosity) contact potential p is the set where the inequality in (2) holds as an equality. We will call it *contact set* and denote it by

$$\Lambda_{\mathsf{p}} := \left\{ (\tau, v, \xi) \in [0,+\infty) \times \mathbb{R}^d \times \mathbb{R}^d \, : \, \mathsf{p}(\tau, v, \xi) = \langle v, \xi \rangle \right\}, \tag{3.15}$$

whereas we will use the notation

$$\Lambda_{\mathsf{p},0} := \Lambda_{\mathsf{p}} \cap \{0\} \times \mathbb{R}^d \times \mathbb{R}^d = \{ (v, \xi) \in \mathbb{R}^d \times \mathbb{R}^d \, : \, \mathsf{p}(0, v, \xi) = \langle v, \xi \rangle \}. \tag{3.16}$$

Let us point out a first important consequence of the properties defining a contact potential:

Lemma 3.2. *For fixed $(\tau, \xi) \in [0,+\infty) \times \mathbb{R}^d$, denote by $\partial_v \mathsf{p}(\tau, \cdot, \xi)(v)$ the (convex analysis) subdifferential at v of the functional $v \mapsto \mathsf{p}(\tau, v, \xi)$. Then,*

$$(\tau, v, \xi) \in \Lambda_{\mathsf{p}} \;\Leftrightarrow\; \xi \in \partial_v \mathsf{p}(\tau, \cdot, \xi)(v) \tag{3.17}$$

Proof. Since $v \mapsto \mathsf{p}(\tau, v, \xi)$ is convex and positively homogeneous of degree 1, we have (cf. (3.5)),

$$\xi \in \partial_v \mathsf{p}(\tau, \cdot, \xi)(v) \quad \text{iff} \quad \begin{cases} \langle \xi, \tilde{v} \rangle \leq \mathsf{p}(\tau, \tilde{v}, \xi) & \text{for all } \tilde{v} \in \mathbb{R}^d, \\ \langle \xi, v \rangle = \mathsf{p}(\tau, v, \xi), \end{cases}$$

and the thesis follows. □

Remark 3.3. Observe that, for fixed $\tau \in [0,+\infty)$, the function $\mathsf{p}(\tau, \cdot, \cdot) : \mathbb{R}^d \times \mathbb{R}^d \to [0,+\infty)$ enjoys some of the properties of the notion of *bipotential* (cf., e.g., [BdV08]), which is by definition a functional $\mathsf{b} : \mathbb{R}^d \times \mathbb{R}^d \to [0,+\infty]$ convex and lower semicontinuous w.r.t. *both* variables, separately, and fulfilling $\mathsf{b}(v, \xi) \geq \langle v, \xi \rangle$ for all $(v, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$, as well as a stronger version of (3.17), namely

$$(v, \xi) \in \Lambda_{\mathsf{b}} \;\Leftrightarrow\; \xi \in \partial_v \mathsf{b}(\cdot, \xi)(v) \;\Leftrightarrow\; v \in \partial_\xi \mathsf{b}(v, \cdot)(\xi) \,,$$

where the contact set Λ_{b} is defined similarly as in (3.15).

As discussed in [MRS12a], the conditions defining the notion of bipotential seem to be too restrictive for the contact potentials arising in the vanishing-viscosity limit of viscous systems approximating rate-independent evolution. Nonetheless, in Sec. 4 we will see how *viscosity contact potentials* can in fact be generated, via Γ-convergence, by bipotentials associated with families of dissipation potentials.

3.2. BV **solutions to rate-independent systems.** We are now in a position to recall the preliminary definitions at the basis of the concept of *Balanced Viscosity* solution; notice that all of them involve the *reduced* contact potential $\mathsf{p}(0,\cdot,\cdot)$ and the energy functional $\mathcal{E} \in \mathrm{C}^1([0,T]\times\mathbb{R}^d)$.

First of all, we introduce the (possibly asymmetric) Finsler distance coming into play in the description of the energetic behaviour of a rate-independent system at a jump time: For a fixed $t \in [0,T]$, the Finsler distance induced by p and $\mathcal{E}$ at the time t is defined for every $u_0,\, u_1 \in \mathbb{R}^d$ by

$$\Delta_{\mathsf{p},\mathcal{E}}(t;u_0,u_1) := \inf\left\{ \int_{r_0}^{r_1} \mathsf{p}(0,\dot\theta(r),-\mathrm{D}\mathcal{E}(t,\theta(r)))\,\mathrm{d}r \ : \ \theta \in \mathrm{AC}([r_0,r_1];\mathbb{R}^d),\ \theta(r_0)=u_0,\ \theta(r_1)=u_1 \right\}. \tag{3.18}$$

Observe that, if p is a Ψ_0-non degenerate contact potential for some 1-positively homogeneous potential Ψ_0, we clearly have $\Delta_{\mathsf{p},\mathcal{E}}(t;u_0,u_1) \geq \Delta_{\Psi_0}(u_0,u_1) := \Psi_0(u_1-u_0)$. The Finsler distances from (3.18) induce a notion of total variation that measures the dissipation of a BV-curve at its jump points, mimicking the notion (3.11) of Ψ_0-total variation. Namely, along the footsteps of [MRS12a, Def. 3.4] and in analogy with (3.14), for a given curve $u \in \mathrm{BV}([0,T];\mathbb{R}^d)$ with jump set J_u, we define the *jump variation* of u induced by $(\mathsf{p},\mathcal{E})$ on an interval $[a,b]\subset[0,T]$ by

$$\begin{aligned}\mathrm{Jmp}_{\mathsf{p},\mathcal{E}}(u;[a,b]) :=& \Delta_{\mathsf{p},\mathcal{E}}(a;u(a),u(a_+)) \\ &+ \sum_{t\in \mathrm{J}_u\cap(a,b)} \left(\Delta_{\mathsf{p},\mathcal{E}}(t;u(t_-),u(t)) + \Delta_{\mathsf{p},\mathcal{E}}(t;u(t),u(t_+))\right) + \Delta_{\mathsf{p},\mathcal{E}}(b;u(b_-),u(b))\,.\end{aligned} \tag{3.19}$$

Finally, given a (non-degenerate) 1-positively homogeneous dissipation potential Ψ_0 and a contact viscosity potential p, the (pseudo-)total variation of a curve $u\in\mathrm{BV}([0,T];\mathbb{R}^d)$ induced by $(\Psi_0,\mathsf{p},\mathcal{E})$ is defined by (cf. (3.13))

$$\mathrm{Var}_{\Psi_0,\mathsf{p},\mathcal{E}}(u;[a,b]) := \mu_{\Psi_0,\mathrm{co}}([a,b]) + \mathrm{Jmp}_{\mathsf{p},\mathcal{E}}(u;[a,b]) \quad \text{for any } [a,b]\subset[0,T], \tag{3.20}$$

with $\mu_{\Psi_0,\mathrm{co}}$ from (3.12) the diffuse part of the total variation measure of the map $t\mapsto \mathrm{Var}_{\Psi_0}(u;[0,t])$. Let us mention that the notation $\mathrm{Var}_{\Psi_0,\mathsf{p},\mathcal{E}}$ is used here with slight abuse, since $\mathrm{Var}_{\Psi_0,\mathsf{p},\mathcal{E}}$ does not enjoy all of the standard properties of total variation functionals, see [MRS12a, Rmk. 3.6] for further details. Also observe that, if p is Ψ_0-non degenerate, then we have $\mathrm{Var}_{\Psi_0,\mathsf{p},\mathcal{E}}(u;[a,b]) \geq \mathrm{Var}_{\Psi_0}(u;[a,b])$.

We are finally in a position to recall the concept of *Balanced Viscosity* solution, cf. [MRS12a, Def. 4.1] and [MRS16, Def. 3.10].

Definition 3.4 (Balanced Viscosity solution)**.** *Given a (non-degenerate)* 1*-homogeneous dissipation potential* Ψ_0 *and a (non-degenerate) viscosity contact potential* p*, we say that a curve* $u\in\mathrm{BV}([0,T];\mathbb{R}^d)$ *is a* Balanced Viscosity (BV) solution to the rate-independent system $(\Psi_0,\mathsf{p},\mathcal{E})$ *if it fulfills the* local stability $(\mathrm{S}_{\mathrm{loc}})$ *and the* $(\mathrm{E}_{\Psi_0,\mathsf{p},\mathcal{E}})$-energy balance

$$-\,\mathrm{D}\mathcal{E}(t,u(t)) \in K^* \quad \textit{for all} \ \ t\in[0,T]\setminus \mathrm{J}_u, \tag{$\mathrm{S}_{\mathrm{loc}}$}$$

$$\mathrm{Var}_{\Psi_0,\mathsf{p},\mathcal{E}}(u;[0,t]) + \mathcal{E}(t,u(t)) = \mathcal{E}(0,u(0)) + \int_0^t \partial_t\mathcal{E}(s,u(s))\,\mathrm{d}s \quad \textit{for all } t\in[0,T] \tag{$\mathrm{E}_{\Psi_0,\mathsf{p},\mathcal{E}}$}$$

with $K^* = \partial\Psi_0(0)$*.*

While referring to [MRS12a, Sec. 4] and [MRS16, Sec. 3] for a detailed survey of the properties of BV solutions, let us only mention here that this concept yields a thorough description of the energetic behavior of the solution at jumps through the concept of *optimal jump transition.* For fixed $t\in[0,T]$ and $u_-,u_+\in\mathbb{R}^d$, we call a curve $\theta\in\mathrm{AC}([0,1];\mathbb{R}^d)$ (up to a rescaling, we may indeed suppose the curves in (3.18) to be defined on $[0,1]$), with $\theta(0)=u_-$ and $\theta(1)=u_+$, a $(\mathsf{p},\mathcal{E}_t)$-optimal transition between u_- and u_+ if

$$\mathcal{E}(t,u_-) - \mathcal{E}(t,u_+) = \Delta_{\mathsf{p},\mathcal{E}}(t;u_-,u_+) = \mathsf{p}(0,\dot\theta(r),-\mathrm{D}\mathcal{E}(t,\theta(r))) > 0 \qquad \text{for a.a. } r\in(0,1). \tag{3.21}$$

The following result subsumes [MRS12a, Prop. 4.6, Thm. 4.7].

Proposition 3.5. *Let $u \in \mathrm{BV}([0,T];\mathbb{R}^d)$ be a Balanced Viscosity solution to the rate-independent system $(\Psi_0, \mathsf{p}, \mathcal{E})$. Then, at every jump time $t \in \mathrm{J}_u$ there exists a $(\mathsf{p}, \mathcal{E}_t)$-optimal transition θ^t between the left and right-limits $u_-(t)$ and $u_+(t)$, such that $\theta^t(r) = u(t)$ for some $r \in [0,1]$. Moreover, any optimal jump transition θ^t between $u_-(t)$ and $u_+(t)$ complies with the contact contact condition*

$$(\dot{\theta}^t(r), -\mathrm{D}\mathcal{E}(t, \theta^t(r))) \in \Lambda_{\mathsf{p},0} \qquad \text{for a.a. } r \in (0,1), \tag{3.22}$$

with $\Lambda_{\mathsf{p},0}$ from (3.16).

A crucial consequence of (3.22) and of (3.17) from Lemma 3.2 is that any optimal jump transition θ^t complies with the subdifferential inclusion

$$-\mathrm{D}\mathcal{E}(t, \theta^t(r)) \in \partial_v \mathsf{p}(0, \cdot, -\mathrm{D}\mathcal{E}(t, \theta^t(r)))(\dot{\theta}^t(r)) \qquad \text{for a.a. } r \in (0,1). \tag{3.23}$$

This explicitly highlights how the contact potential p enters into the description of the solution behavior at jumps.

With the last result of this section we reformulate the BV solution concept in terms of the null-minimization of a functional defined on BV-trajectories; this will be crucial for the variational convergence analysis developed in Sec. 5. Namely, given a rate-independent system $(\Psi_0, \mathsf{p}, \mathcal{E})$, we define the functional $\mathscr{J}_{\Psi_0,\mathsf{p},\mathcal{E}} : \mathrm{BV}([0,T];\mathbb{R}^d) \to (-\infty, +\infty]$ by

$$\begin{aligned} \mathscr{J}_{\Psi_0,\mathsf{p},\mathcal{E}}(u) :&= \mathrm{Var}_{\Psi_0,\mathsf{p},\mathcal{E}}(u;[0,T]) + \int_0^T \Psi_0^*(-\mathrm{D}\mathcal{E}(t,u(t)))\,\mathrm{d}t + \mathcal{E}(T,u(T)) - \mathcal{E}(0,u(0)) - \int_0^T \partial_t \mathcal{E}(s,u(s))\,\mathrm{d}s \\ &= \int_0^T \Psi_0(\dot{u}(s)) + \Psi_0^*(-\mathrm{D}\mathcal{E}(s,u(s)))\,\mathrm{d}s + \mu_{\Psi_0,\mathrm{C}}([0,T]) + \mathrm{Jmp}_{\mathsf{p},\mathcal{E}}(u;[0,T]) \\ &\qquad + \mathcal{E}(T,u(T)) - \mathcal{E}(0,u(0)) - \int_0^T \partial_t \mathcal{E}(s,u(s))\,\mathrm{d}s. \end{aligned} \tag{3.24}$$

We then have the following

Proposition 3.6. *A curve $u \in \mathrm{BV}([0,T];\mathbb{R}^d)$ is a Balanced Viscosity solution to the rate-independent system $(\Psi_0, \mathsf{p}, \mathcal{E})$ if and only if*

$$0 = \mathscr{J}_{\Psi_0,\mathsf{p},\mathcal{E}}(u) \le \mathscr{J}_{\Psi_0,\mathsf{p},\mathcal{E}}(v) \qquad \text{for all } v \in \mathrm{BV}([0,T];\mathbb{R}^d) \tag{3.25}$$

Proof. First of all, observe that conditions $(\mathrm{S_{loc}})$–$(\mathrm{E}_{\Psi_0,\mathsf{p},\mathcal{E}})$ are indeed equivalent to $(\mathrm{S}'_{\mathrm{loc}})$–$(\mathrm{E}_{\Psi_0,\mathsf{p},\mathcal{E}})$, with

$$-\mathrm{D}\mathcal{E}(t,u(t)) \in K^* \quad \text{for } \mathscr{L}^1-\text{a.a. } t \in (0,T). \tag{$\mathrm{S}'_{\mathrm{loc}}$}$$

Indeed, if $(\mathrm{S}'_{\mathrm{loc}})$ holds, with a continuity argument one deduces $-\mathrm{D}\mathcal{E}(t,u(t)) \in K^*$ at all $t \in [0,T] \setminus \mathrm{J}_u$.

Clearly, $(\mathrm{S}'_{\mathrm{loc}})$–$(\mathrm{E}_{\Psi_0,\mathsf{p},\mathcal{E}})$ are then equivalent to

$$\mathscr{J}_{\Psi_0,\mathsf{p},\mathcal{E}}(u) = 0. \tag{3.26}$$

Now, with an argument based on the chain rule for $\mathcal{E}$, one sees (cf. the proof of [MRS16, Cor. 3.4]) that along a given curve $v \in \mathrm{BV}([0,T];\mathbb{R}^d)$ the map $\mathscr{J}_{\Psi_0,\mathsf{p},\mathcal{E}}(v) \ge 0$, so that (3.26) holds if and only if $\mathscr{J}_{\Psi_0,\mathsf{p},\mathcal{E}}(u) \le 0$, i.e. $u \in \mathrm{Argmin}_{v \in \mathrm{BV}([0,T];\mathbb{R}^d)} \mathscr{J}_{\Psi_0,\mathsf{p},\mathcal{E}}(v)$. This concludes the proof. □

4. Generation of viscosity contact potentials via Γ-convergence

In this section we show a possible way to generate a viscosity contact potential via a Γ-convergence procedure, starting from a family $(\Psi_n)_n$ of dissipation potentials with *superlinear growth at infinity* (cf. (3.2)).

Preliminarily, given a convex dissipation potential Ψ, we define the *bipotential* $\mathsf{b}_\Psi : [0,+\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0,+\infty]$ induced by Ψ via

$$\mathsf{b}_\Psi(\tau,v,\xi) := \begin{cases} \tau\Psi\left(\frac{v}{\tau}\right) + \tau\Psi^*(\xi) & \text{for } \tau>0, \\ 0 & \text{for } \tau=0,\ v=0, \\ +\infty & \text{for } \tau=0 \text{ and } v\neq 0, \end{cases} = \begin{cases} \tau\Psi\left(\frac{v}{\tau}\right) + \tau\Psi^*(\xi) & \text{for } \tau>0, \\ I_{\{0\}}(v) & \text{for } \tau=0. \end{cases} \tag{4.1}$$

It is immediate to check that

(1) for every $(v,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$ the map $\tau \mapsto \mathsf{b}_\Psi(\tau,v,\xi)$ is convex;
(2) for every $\tau \geq 0$ the functional $(v,\xi) \mapsto \mathsf{b}_\Psi(\tau,v,\xi)$ is a bipotential in the sense of [BdV08] (cf. Remark 3.3);
(3) for every $v \neq 0$ and $\xi \in \mathbb{R}^d$ with $\Psi^*(\xi) \neq 0$, the set $\mathrm{Argmin}_{\tau>0}\mathsf{b}_\Psi(\tau,v,\xi)$ is non-empty,

where the latter property is due to the fact that $\lim_{\tau\downarrow 0}\mathsf{b}_\Psi(\tau,v,\xi) = +\infty$ due to the superlinear growth of Ψ, and $\lim_{\tau\uparrow+\infty}\mathsf{b}_\Psi(\tau,v,\xi) = +\infty$.

Let us now be given a sequence $(\Psi_n)_n$ of dissipation potentials, and let $(\mathsf{b}_{\Psi_n})_n$ be the associated bipotentials. We assume the following.

Hypothesis 4.1. Let $\mathsf{p} : [0,+\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0,+\infty]$ be defined by

$$\mathsf{p} = \Gamma\text{-}\liminf_n \mathsf{b}_{\Psi_n} \quad \text{i.e.} \quad \mathsf{p}(\tau,v,\xi) := \inf\{\liminf_{n\to\infty} \mathsf{b}_{\Psi_n}(\tau_n,v_n,\xi_n) \ : \ \tau_n\to\tau, \quad v_n\to v \quad \xi_n\to\xi\}. \tag{4.2}$$

Then,

$$\begin{aligned} &\text{for every } \xi\in\mathbb{R}^d \text{ there exists } (\xi_n)_n\subset\mathbb{R}^d \text{ with } \xi_n\to\xi \text{ and } \mathsf{p}(\cdot,\cdot,\xi) = \Gamma\text{-}\limsup_{n\to\infty}\mathsf{b}_{\Psi_n}(\cdot,\cdot,\xi_n) \qquad \text{i.e.} \\ &\mathsf{p}(\tau,v,\xi) = \inf_{(\xi_n)_n\subset\mathbb{R}^d,\,\xi_n\to\xi}\left\{\limsup_{n\to\infty}\mathsf{b}_{\Psi_n}(\tau_n,v_n,\xi_n) \ : \ \tau_n\to\tau, \quad v_n\to v\right\}. \end{aligned} \tag{4.3}$$

In Section 5.2 ahead, we will exhibit two classes of dissipations potentials $(\Psi_n)_n$, with superlinear growth at infinity, and associated functionals p, complying with Hypothesis 4.1.

Observe that with (4.3) we are imposing a stronger condition than $\mathsf{p} = \Gamma\text{-}\limsup_{n\to\infty}\mathsf{b}_{\Psi_n}$, namely we are asking that

$$\forall\,\xi\in\mathbb{R}^d \ \ \exists\,(\xi_n)_n\subset\mathbb{R}^d \ : \ \ \forall\,(\tau,v)\in[0,+\infty)\times\mathbb{R}^d \ \exists\,(\tau_n,v_n)_n \ \text{ s.t. } \begin{cases} \tau_n\to\tau, \\ v_n\to v, \\ \limsup_{n\to\infty}\mathsf{b}_{\Psi_n}(\tau_n,v_n,\xi_n)\leq\mathsf{p}(\tau,v,\xi)\,. \end{cases} \tag{4.4}$$

This property will play a key role in the proof of Lemma 4.3 below.

The main result of this section ensures that the functional p generated via (4.2)–(4.3) is a contact potential in the sense of Definition 3.1.

Theorem 4.2. *Let $(\Psi_n)_n$ be a sequence of dissipation potentials on $\mathbb{R}^d$ complying with Hypothesis 4.1. Then, p is a viscosity contact potential according to Def. 3.1, and there exists a 1-homogeneous dissipation potential Ψ_0 such that*

$$\mathsf{p}(\tau,v,\xi)\geq\Psi_0(v) \qquad \textit{for all } (\tau,v,\xi)\in[0,+\infty)\times\mathbb{R}^d\times\mathbb{R}^d. \tag{4.5}$$

Moreover, if the dissipation potentials $(\Psi_n)_n$ fulfill

$$\exists\,M>0,\ (M_n)_n\subset(0,+\infty) \ \textit{s.t.} \ M_n\to 0 \ \textit{and} \ \forall\,n\in\mathbb{N} \ \ \forall\,v\in\mathbb{R}^d \ \textit{there holds} \qquad \Psi_n(v)\geq M\|v\|-M_n, \tag{4.6}$$

then Ψ_0 is non-degenerate, and thus p is Ψ_0-non degenerate.

We postpone the proof of Theorem 4.2 to the end of this section, after obtaining a series of preliminary lemmas on the structure that p defined by Hypothesis 4.1 inherits from the potentials Ψ_n.

Lemma 4.3. *Assume Hypothesis 4.1. Then, for every $(\tau,v,\xi)\in[0,+\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ there holds*

(1) $\mathsf{p}(\tau, v, \xi) \geq \langle v, \xi \rangle$*;*
(2) the map $(\tau, v) \mapsto \mathsf{p}(\tau, v, \xi)$ *is convex and positively homogeneous of degree* 1.

Proof. Property (1) is an immediate consequence of (4.2), using that for every $n \in \mathbb{N}$ there holds $\mathsf{b}_{\Psi_n}(\tau, v, \xi) \geq \langle v, \xi \rangle$ for every $(\tau, v, \xi) \in [0, +\infty) \times \mathbb{R}^d \times \mathbb{R}^d$.

As for (2), for fixed ξ let $(\xi_n)_n$ fulfill (4.3). For fixed (τ_0, v_0) and (τ_1, v_1) let $(\tau_n^i, v_n^i)_n$, $i = 1, 2$, be two associated recovery sequences for $\mathsf{b}_{\Psi_n}(\cdot, \cdot, \xi_n)$ as in (4.4). Then, for every $\lambda \in [0, 1]$ there holds

$$\begin{aligned} \mathsf{p}((1-\lambda)\tau_0 + \lambda\tau_1, (1-\lambda)v_0 + \lambda v_1, \xi) &\overset{(1)}{\leq} \liminf_{n\to\infty} \mathsf{b}_{\Psi_n}((1-\lambda)\tau_n^0 + \lambda\tau_n^1, (1-\lambda)v_n^0 + \lambda v_n^1, \xi_n) \\ &\overset{(2)}{\leq} \limsup_{n\to\infty} (1-\lambda)\mathsf{b}_{\Psi_n}(\tau_n^0, v_n^0, \xi_n) + \lambda \mathsf{b}_{\Psi_n}(\tau_n^1, v_n^1, \xi_n) \\ &\overset{(3)}{\leq} (1-\lambda)\mathsf{p}(\tau_0, v_0, \xi) + \lambda \mathsf{p}(\tau_1, v_1, \xi), \end{aligned}$$

where (1) follows from (4.2), (2) from the convexity of the maps $\mathsf{b}_{\Psi_n}(\cdot, \cdot, \xi_n)$, and (3) from (4.3).

With an analogous argument one proves that $\mathsf{p}(\cdot, \cdot, \xi)$ is 1-positively homogeneous. □

We now show that, for $\tau > 0$ the functional $\mathsf{p}(\tau, \cdot, \cdot)$ has the same form (4.1) as $\mathsf{b}_{\Psi_n}(\tau, \cdot, \cdot)$, cf. (4.8).

Lemma 4.4. *Assume Hypothesis 4.1. Let* $\Psi_0 : \mathbb{R}^d \to [0 + \infty)$ *be defined by*

$$\Psi_0(v) := \mathsf{p}(1, v, 0). \tag{4.7}$$

Then, Ψ_0 *is a* 1*-positively homogeneous dissipation potential, the sequence* $(\Psi_n)_n$ Γ*-converges to* Ψ_0*, and thus* $(\Psi_n^*)_n$ Γ*-converges to* Ψ_0^**. Furthermore,*

$$\mathsf{p}(\tau, v, \xi) = \tau \Psi_0\left(\frac{v}{\tau}\right) + \tau \Psi_0^*(\xi) \quad \textit{for all } \tau > 0 \textit{ and all } (v, \xi) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{4.8}$$

Proof. Observe that Ψ_0 from (4.7) is convex and 1-homogeneous thanks to Lemma 4.3. It follows from (4.2) and (4.3), applied with the choices $\tau = 1$ and $\xi = 0$, that $\Psi_0 = \Gamma\text{-}\lim_{n\to\infty} \Psi_n$. Then, $(\Psi_n^*)_n$ Γ-converges to Ψ_0^* by [Att84, Thm. 2.18, p. 495]. As a consequence of these convergences and of (4.1), we have (4.8). □

Our next two results address the characterization of p for $\tau = 0$, providing a formula for $\mathsf{p}(0, v, w)$ in the two cases $\Psi_0^*(\xi) < +\infty$ and $\Psi_0^*(\xi) = +\infty$.

Lemma 4.5. *Assume Hypothesis 4.1. If* $\Psi_0^*(\xi) < +\infty$*, then*

$$\mathsf{p}(0, v, \xi) = \liminf_{\tau\to 0} \tau \Psi_0\left(\frac{v}{\tau}\right) = \Psi_0(v) \qquad \textit{for all } v \in \mathbb{R}^d. \tag{4.9}$$

Proof. It follows from (4.8) and the fact that $\Psi_0^*(\xi) < +\infty$ that

$$\mathsf{p}(0, v, \xi) \leq \liminf_{\tau\to 0} \mathsf{p}(\tau, v, \xi) \leq \liminf_{\tau\to 0} \tau \Psi_0\left(\frac{v}{\tau}\right). \tag{4.10}$$

To prove the converse inequality, we preliminarily observe that, for any dissipation potential Ψ, for every $v \in \mathbb{R}^d$ the map $\tau \mapsto \Psi\left(\frac{v}{\tau}\right)$ is non-increasing. Therefore for all $0 < \tau < \sigma < 1$ we have

$$\tau \Psi\left(\frac{v}{\tau}\right) \geq \sigma \Psi\left(\frac{v}{\sigma}\right). \tag{4.11}$$

Now, let us fix a sequence $\xi_n \to \xi$ for which (4.3) holds, and accordingly a sequence $(\tau_n, v_n) \to (0, v)$ such that $\mathsf{p}(0, v, \xi) = \liminf_{n\to\infty}(\tau_n \Psi_n(v_n/\tau_n) + \tau_n \Psi_n^*(\xi_n))$. It follows from inequality (4.11) applied to the functionals Ψ_n that for every $\sigma \in (0, 1)$

$$\liminf_{n\to\infty} \left(\tau_n \Psi_n\left(\frac{v_n}{\tau_n}\right) + \tau_n \Psi_n^*(\xi_n)\right) \geq \liminf_{n\to\infty} \left(\sigma \Psi_n\left(\frac{v_n}{\sigma}\right)\right) = \sigma \Psi_0\left(\frac{v}{\sigma}\right),$$

where we have also exploited the positivity of the functionals Ψ_n^*. Therefore, in view of (4.3) we find

$$\mathsf{p}(0, v, \xi) \geq \sigma \Psi_0\left(\frac{v}{\sigma}\right)$$

and conclude the converse of (4.10) passing to the limit as $\sigma \to 0$. $\square$

Lemma 4.6. *Assume Hypothesis 4.1. If $\Psi_0^*(\xi) = +\infty$, then*

$$\mathsf{p}(0, v, \xi) = \Gamma\text{-}\liminf_{n\to\infty} \inf_{\tau>0} \mathsf{b}_{\Psi_n}(\tau, v, \xi) \qquad \textit{for all } v \in \mathbb{R}^d. \tag{4.12}$$

Proof. Inequality $\geq$ follows from the definition of p. To prove the converse one, we may suppose that $v \neq 0$, since $\mathsf{p}(0, 0, \xi) = 0$. Take $(v_n, \xi_n) \to (v, \xi)$ that attains $\Gamma\text{-}\liminf_{n\to\infty} \inf_{\tau>0} \mathsf{b}_{\Psi_n}(\tau, v, \xi)$, i.e. $\inf_{\tau>0} \mathsf{b}_{\Psi_n}(\tau, v_n, \xi_n) \to \Gamma\text{-}\liminf_{n\to\infty} \inf_{\tau>0} \mathsf{b}_{\Psi_n}(\tau, v, \xi)$. In particular, $\liminf_{n\to\infty} \Psi_n^*(\xi_n) = +\infty$. Therefore, we may choose $\bar\tau_n$ as

$$\bar\tau_n \in \mathrm{Argmin}_{\tau>0} \left(\tau \Psi_n \left(\frac{v_n}{\tau} \right) + \tau \Psi_n^*(\xi_n) \right).$$

Since $\liminf_{n\to\infty} \Psi_n^*(\xi_n) = +\infty$, it is clear that $\bar\tau_n \to 0$, hence

$$\Gamma\text{-}\liminf_{n\to\infty} \inf_{\tau>0} \mathsf{b}_{\Psi_n}(\tau, v, \xi) = \lim_{n\to\infty} \left(\bar\tau_n \Psi_n \left(\frac{v_n}{\bar\tau_n} \right) + \bar\tau_n \Psi_n^*(\xi_n) \right) \geq \mathsf{p}(0, v, \xi)$$

thanks to (4.2). $\square$

We now prove a pseudo-monotonicity result for p.

Lemma 4.7. *Assume Hypothesis 4.1. Then, for every τ, $\bar\tau \in [0, +\infty)$, v, $\bar v \in \mathbb{R}^d$ and ξ, $\bar\xi \in \mathbb{R}^d$ we have that*

$$\Big(\mathsf{p}(\tau, v, \xi) - \mathsf{p}(\tau, v, \bar\xi) \Big) \Big(\mathsf{p}(\bar\tau, \bar v, \xi) - \mathsf{p}(\bar\tau, \bar v, \bar\xi) \Big) \geq 0. \tag{4.13}$$

Proof. Observe that (4.13) holds for the bipotentials b_{Ψ_n}: indeed, in that case it reduces to $\tau\bar\tau(\Psi_n^*(\xi) - \Psi_n^*(\bar\xi))^2 \geq 0$.

Assume that $\mathsf{p}(\tau, v, \xi) > \mathsf{p}(\tau, v, \bar\xi)$ and choose $\bar\xi_n$ as in (4.3) with (τ_n, v_n) such that

$$(\tau_n, v_n, \bar\xi_n) \to (\tau, v, \bar\xi), \qquad \mathsf{b}_{\Psi_n}(\tau_n, v_n, \bar\xi_n) \to \mathsf{p}(\tau, v, \bar\xi). \tag{4.14}$$

It follows from the definition (4.2) of p that $\mathsf{p}(\tau, v, \xi) \leq \liminf_{n\to\infty} \mathsf{b}_{\Psi_n}(\tau_n, v_n, \xi_n)$ for every sequence $\xi_n \to \xi$ in $\mathbb{R}^d$, and for (τ_n, v_n) as in (4.14). Then

$$0 < \mathsf{p}(\tau, v, \xi) - \mathsf{p}(\tau, v, \bar\xi) \leq \liminf_{n\to\infty} \Big(\mathsf{b}_{\Psi_n}(\tau_n, v_n, \xi_n) - \mathsf{b}_{\Psi_n}(\tau_n, v_n, \bar\xi_n) \Big). \tag{4.15}$$

Therefore, for sufficiently big n we have that

$$\mathsf{b}_{\Psi_n}(\tau_n, v_n, \xi_n) - \mathsf{b}_{\Psi_n}(\tau_n, v_n, \bar\xi_n) \geq 0. \tag{4.16}$$

Now, again in view of (4.3), choose $\xi_n \to \xi$ (notice that (4.15) holds for *any* sequence ξ_n converging to ξ) and $\bar\tau_n \to \bar\tau$, $\bar v_n \to \bar v$ such that $\limsup_{n\to\infty} \mathsf{b}_{\Psi_n}(\bar\tau_n, \bar v_n, \xi_n) \leq \mathsf{p}(\bar\tau, \bar v, \xi)$. Since $\liminf_{n\to\infty} \mathsf{b}_{\Psi_n}(\bar\tau_n, \bar v_n, \bar\xi_n) \geq \mathsf{p}(\bar\tau, \bar v, \bar\xi)$ by (4.2), we conclude that

$$\mathsf{p}(\bar\tau, \bar v, \xi) - \mathsf{p}(\bar\tau, \bar v, \bar\xi) \geq \limsup_{n\to\infty} \big(\mathsf{b}_{\Psi_n}(\bar\tau_n, \bar v_n, \xi_n) - \mathsf{b}_{\Psi_n}(\bar\tau_n, \bar v_n, \bar\xi_n) \big) \geq 0,$$

taking into account that $\mathsf{b}_{\Psi_n}(\bar\tau_n, \bar v_n, \xi_n) - \mathsf{b}_{\Psi_n}(\bar\tau_n, \bar v_n, \bar\xi_n) \geq 0$ for sufficiently big n thanks to (4.16) and the previously observed monotonicity property (4.13) for b_{Ψ_n}. Thus, (4.13) follows. $\square$

Finally, let us consider contact sets associated with the bipotentials b_{Ψ_n}, i.e.

$$\Lambda_{\mathsf{b}_{\Psi_n}} := \left\{ (\tau, v, \xi) \in [0, +\infty) \times \mathbb{R}^d \times \mathbb{R}^d \, : \langle v, \xi \rangle = \mathsf{b}_{\Psi_n}(\tau, v, \xi) \right\}.$$

Observe that for every $n \in \mathbb{N}$

(1) $\Lambda_{\mathsf{b}_{\Psi_n}} \cap \{0\} \times \mathbb{R}^d \times \mathbb{R}^d = \{0\} \times \{0\} \times \mathbb{R}^d$;
(2) for $\tau > 0$, if $(\tau, v, \xi) \in \Lambda_{\mathsf{b}_{\Psi_n}}$, then $\tau \in \mathrm{Argmin}_{\sigma\in(0,+\infty)}(\sigma\Psi_n(\frac{v}{\sigma}) + \sigma\Psi_n^*(\xi))$.

The following closedness property may be easily derived from (4.2).

Lemma 4.8. *Assume Hypothesis 4.1. Then,*

$$\begin{cases} (\tau_n, v_n, \xi_n) \in \Lambda_{\mathsf{b}_{\Psi_n}}, \\ (\tau_n, v_n, \xi_n) \to (\tau, v, \xi) \end{cases} \quad \Rightarrow \quad (\tau, v, \xi) \in \Lambda_{\mathsf{p}}. \tag{4.17}$$

We are now in a position to carry out the **proof of Theorem 4.2** by verifying that p complies with properties (1)–(5) from Definition 3.1.

Properties (1)&(2) are guaranteed by Lemma 4.3, whereas (3) ensues from (4.8) in Lemma 4.4. Concerning property (5), observe that (4.5) ensues from (4.8) for $\tau > 0$. For $\tau = 0$, it directly follows from (4.9) in the case $\Psi_0^*(\xi) < +\infty$, whereas for $\Psi_0^*(\xi) = +\infty$ we use the monotonicity property (4.13), giving

$$(\mathsf{p}(1, v, \xi) - \mathsf{p}(1, v, 0))(\mathsf{p}(0, v, \xi) - \mathsf{p}(0, v, 0)) \geq 0.$$

Now, $\mathsf{p}(1, v, \xi) = \Psi_0(v) + \Psi_0^*(\xi) = +\infty$, hence we deduce that $\mathsf{p}(0, v, \xi) \geq \mathsf{p}(0, v, 0) \geq \Psi_0(v)$ (here we have used that $\Psi_0^*(0) = 0$).

Under the additional (4.6), it is immediate to check that Ψ_0 given by (4.7) is non-degenerate, i.e. property (4). This concludes the proof of Thm. 4.2. ∎

5. **Main results**

Let us consider a sequence $(\Psi_n)_n$ of dissipation potentials on $\mathbb{R}^d$ with *superlinear growth at infinity*, namely fulfilling (3.2) for every $n \in \mathbb{N}$. It follows from [MRS13], extending the classical results by Colli&Visintin (cf. [CV90, Col92]) that for every $n \in \mathbb{N}$ there exists at least a solution $u \in \mathrm{AC}([0,T]; \mathbb{R}^d)$ of the Cauchy problem for the generalized gradient system $(\Psi_n, \mathcal{E})$, with $\mathcal{E}$ complying with (E). Namely, u solves the doubly nonlinear differential inclusion

$$\begin{cases} \partial \Psi_n(\dot{u}(t)) + \mathrm{D}\mathcal{E}(t, u(t)) \ni 0 & \text{for a.a. } t \in (0,T), \\ u(0) = u_0 \end{cases} \tag{5.1}$$

for a given datum $u_0 \in \mathbb{R}^d$. Furthermore, again an argument (also often referred to as *De Giorgi principle*, see [Mie16]) based on the chain rule, cf. [MRS13], shows that a curve $u \in \mathrm{AC}([0,T]; \mathbb{R}^d)$ is a solution of the gradient system $(\Psi_n, \mathcal{E})$ if and only if it is a null-minimizer for the (positive) functional of trajectories $\mathcal{J}_{\Psi_n, \mathcal{E}} : \mathrm{AC}([0,T]; \mathbb{R}^d) \to [0, +\infty)$ defined by

$$\mathcal{J}_{\Psi_n, \mathcal{E}}(u) := \int_0^T (\Psi_n(\dot{u}(s)) + \Psi_n^*(-\mathrm{D}\mathcal{E}(s, u(s))))\, \mathrm{d}s + \mathcal{E}(T, u_n(T)) - \mathcal{E}(0, u_n(0)) - \int_0^T \partial_t \mathcal{E}(s, u_n(s))\, \mathrm{d}s. \tag{5.2}$$

Observe that the positivity of $\mathcal{J}_{\Psi_n, \mathcal{E}}$ follows from

$$\begin{aligned} \int_0^T (\Psi_n(\dot{u}(s)) + \Psi_n^*(-\mathrm{D}\mathcal{E}(s, u(s))))\, \mathrm{d}s &\geq - \int_0^T \langle \mathrm{D}\mathcal{E}(s, u(s)), \dot{u}(s) \rangle \, \mathrm{d}s \\ &= \mathcal{E}(0, u_n(0)) - \mathcal{E}(T, u_n(T)) + \int_0^T \partial_t \mathcal{E}(s, u_n(s))\, \mathrm{d}s, \end{aligned}$$

the last equality by the chain rule.

The **main results of this paper, Theorems 5.2 and 5.8** ahead, concern the Mosco-convergence to the functional $\mathscr{J}_{\Psi_0, \mathsf{p}, \mathcal{E}}$ from (3.24), with respect to the weak-strict topology of $\mathrm{BV}([0,T]; \mathbb{R}^d)$, of a family of functionals $(\mathscr{J}_{\Psi_n, \mathcal{E}})_n$ suitably extending $\mathcal{J}_{\Psi_n, \mathcal{E}}$ to $\mathrm{BV}([0,T]; \mathbb{R}^d)$. Namely, we define

$$\mathscr{J}_{\Psi_n, \mathcal{E}} : \mathrm{BV}([0,T]; \mathbb{R}^d) \to [0, +\infty] \quad \text{by} \quad \mathscr{J}_{\Psi_n, \mathcal{E}}(u) := \begin{cases} \mathcal{J}_{\Psi_n, \mathcal{E}}(u) & \text{if } u \in \mathrm{AC}([0,T]; \mathbb{R}^d), \\ +\infty & \text{otherwise.} \end{cases} \tag{5.3}$$

5.1. **The Γ-liminf result.** First of all, let us fix the compactness properties of a sequence $(u_n)_n \subset \mathrm{BV}([0,T];\mathbb{R}^d)$ with $\sup_n \mathscr{J}_{\Psi_n,\mathcal{E}}(u_n) \le C$, assuming that the potentials Ψ_n comply with a suitable coercivity property.

Proposition 5.1. *Let $(\Psi_n)_n$ be a family of dissipation potentials with superlinear growth at infinity and assume that*

$$\exists\, M_1,\, M_2 > 0 \ \ \forall\, n \in \mathbb{N} \ \ \forall\, v \in \mathbb{R}^d : \quad \Psi_n(v) \ge M_1 \|v\|_1 - M_2. \tag{5.4}$$

Let $(u_n)_n \subset \mathrm{BV}([0,T];\mathbb{R}^d)$ fulfill $\|u_n(0)\| + \mathscr{J}_{\Psi_n,\mathcal{E}}(u_n) \le C$ for some constant $C > 0$ uniform w.r.t. $n \in \mathbb{N}$. Then, there exist a subsequence $k \mapsto n_k$ and a curve u such that $u_{n_k} \rightharpoonup u$ in $\mathrm{BV}([0,T];\mathbb{R}^d)$.

We are now in a position to state the Γ-$\liminf$ result for the sequence $(\mathscr{J}_{\Psi_n,\mathcal{E}})_n$. Its proof is postponed to Section 6.

Theorem 5.2. *Let $(\Psi_n)_n$ be a family of dissipation potentials with superlinear growth at infinity such that the associated bipotentials $(\mathsf{b}_{\Psi_n})_n$ comply with Hypothesis 4.1, with limiting viscosity contact potential p. Let Ψ_0 be the 1-positively homogeneous dissipation potential defined by $\Psi_0(v) := \mathsf{p}(1,v,0)$, and suppose that Ψ_0 is non-degenerate.*

Then, for every $(u_n)_n$, $u \in \mathrm{BV}([0,T];\mathbb{R}^d)$ we have that

$$u_n \rightharpoonup u \ \text{in}\ \mathrm{BV}([0,T];\mathbb{R}^d) \quad \Rightarrow \quad \liminf_{n\to\infty} \mathscr{J}_{\Psi_n,\mathcal{E}}(u_n) \ge \mathscr{J}_{\Psi_0,\mathsf{p},\mathcal{E}}(u). \tag{5.5}$$

More precisely, we have as $n \to \infty$

$$\mathcal{E}(t,u_n(t)) \to \mathcal{E}(t,u(t)) \ \text{and} \ \int_0^t \partial_t \mathcal{E}(r,u_n(r))\,\mathrm{d}r \to \int_0^t \partial_t \mathcal{E}(r,u(r))\,\mathrm{d}r \quad \text{for every } t \in [0,T], \tag{5.6}$$

$$\liminf_{n\to\infty} \int_s^t \left(\Psi_n(\dot u_n(r)) + \Psi_n^*(-\mathrm{D}\mathcal{E}(r,u_n(r)))\right)\,\mathrm{d}r \ge \mathrm{Var}_{\Psi_0,\mathsf{p},\mathcal{E}}(u;[s,t]) + \int_s^t \Psi_0^*(-\mathrm{D}\mathcal{E}(t,u(r)))\,\mathrm{d}r \tag{5.7}$$

for every $0 \le s \le t \le T$.

Note that a sufficient condition for Ψ_0 to be non-degenerate is that the potentials Ψ_n comply with (4.6), cf. Theorem 4.2. A straightforward consequence of Thm. 5.2 is the following result.

Corollary 5.3. *Under the assumptions of Theorem 5.2, let $(u_n)_n \subset \mathrm{AC}([0,T];\mathbb{R}^d)$ fulfill $\mathscr{J}_{\Psi_n}(u_n) \le \varepsilon_n$ for every $n \in \mathbb{N}$, for some vanishing sequence $(\varepsilon_n)_n$.*

Then, any limit point u of $(u_n)_n$ with respect to the weak-$\mathrm{BV}([0,T];\mathbb{R}^d)$-topology is a Balanced Viscosity solution to the rate-independent system $(\Psi_0,\mathsf{p},\mathcal{E})$, and, up to a subsequence, convergences (5.6) and

$$\lim_{n\to\infty} \int_s^t \left(\Psi_n(\dot u_n(r)) + \Psi_n^*(-\mathrm{D}\mathcal{E}(r,u_n(r)))\right)\,\mathrm{d}r = \mathrm{Var}_{\Psi_0,\mathsf{p},\mathcal{E}}(u;[s,t]) + \int_s^t \Psi_0^*(-\mathrm{D}\mathcal{E}(t,u(r)))\,\mathrm{d}r \tag{5.8}$$

hold for all $0 \le s \le t \le T$.

5.2. **Examples.** We now focus on two classes of dissipations potentials $(\Psi_n)_n$, with superlinear growth at infinity, approximating a 1-positively homogeneous dissipation potential Ψ_0. In the first case, the dissipation potentials Ψ_n are obtained by rescaling from a given dissipation potential Ψ with superlinear growth at infinity, and suitably converge to some Ψ_0. In the second case, we consider the stochastic model introduced in Section 2 and the associated potentials Ψ_n given by (2.5): the limiting potential is $\Psi_0(v) = A\|v\|_1$. We will show that, in both cases Hypothesis 4.1 is fulfilled.

The vanishing-viscosity approximation. We consider the dissipation potentials

$$\Psi_n(v) = \Psi_{\varepsilon_n}(v) = \frac{1}{\varepsilon_n}\Psi(\varepsilon_n v) \quad \text{for all } v \in \mathbb{R}^d, \text{ with } \varepsilon_n \downarrow 0, \tag{5.9a}$$

with $\Psi : \mathbb{R}^d \to [0, +\infty)$ a fixed potential with superlinear growth at infinity. We suppose that there exists a 1-homogeneous dissipation potential Ψ_0 such that

$$\Psi_0(v) = \lim_{n\to\infty} \Psi_n(v) = \lim_{n\to\infty} \frac{1}{\varepsilon_n}\Psi(\varepsilon_n v) \quad \text{for all } v \in \mathbb{R}^d. \tag{5.9b}$$

Example 5.4. In particular, we focus on these two cases (cf. [MRS12a, Ex. 2.3]):

(1) **Ψ_0-viscosity:** the superlinear dissipation potential Ψ is obtained augmenting Ψ_0 by a superlinear function of Ψ_0 itself. Namely, given a convex superlinear function $F_V : [0, +\infty) \to [0, +\infty)$, we set

$$\Psi(v) := \Psi_0(v) + F_V(\Psi_0(v)), \quad \text{whence } \Psi_n(v) = \Psi_0(v) + \frac{1}{\varepsilon_n} F_V(\varepsilon_n \Psi_0(v)) \quad \text{for all } v \in \mathbb{R}^d. \tag{5.10}$$

To fix ideas, we may think of $\Psi_0(v) = A\|v\|_1$ and $F_V(\rho) = \frac{1}{2}\rho^2$, giving rise to

$$\Psi_n(v) = A\|v\|_1 + \frac{\varepsilon_n}{2} A^2 \|v\|_1^2. \tag{5.11}$$

(2) **2-norm vanishing-viscosity:** Let us now consider a norm $\|\cdot\|$ on $\mathbb{R}^d$, different from that associated with Ψ_0. We set

$$\Psi(v) := \Psi_0(v) + F_V(\|v\|), \quad \text{whence } \Psi_n(v) = \Psi_0(v) + \frac{1}{\varepsilon_n} F_V(\varepsilon_n \|v\|) \quad \text{for all } v \in \mathbb{R}^d, \tag{5.12}$$

with again $F_V : [0, +\infty) \to [0, +\infty)$ convex and superlinear. In this way we generate, for example, the dissipation potentials

$$\Psi_n(v) = A\|v\|_1 + \frac{\varepsilon_n}{2}\|v\|_2^2, \tag{5.13}$$

with $\|v\|_2 := \left(\sum_{i=1}^d |v_i|^2\right)^{1/2}$.

This family of dissipation potentials comply with the hypotheses of Thm. 5.2.

Proposition 5.5. *The dissipation potentials from* (5.9) *comply with* (4.6) *and with Hypothesis 4.1, where*

$$\mathsf{p} : [0, +\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty] \qquad \textit{is given by } \mathsf{p}(\tau, v, \xi) := \begin{cases} \Psi_0(v) + I_{K^*}(\xi) & \textit{if } \tau > 0, \\ \inf_{\varepsilon_n > 0} \left(\Psi_{\varepsilon_n}(v) + \Psi^*_{\varepsilon_n}(\xi)\right) & \textit{if } \tau = 0. \end{cases} \tag{5.14}$$

The *proof* can be straightforwardly retrieved from the argument for [MRS12a, Lemma 6.1].

Example 5.6 (Example 5.4 continued)**.** Following [MRS12a, Rem. 3.1], we explicitly calculate $\mathsf{p}(0, v, \xi)$, using formula (5.14), in the two cases of Example 5.4:

(1) **Ψ_0-viscosity:** We have

$$\mathsf{p}(0, v, \xi) := \begin{cases} \Psi_0(v) & \text{if } \xi \in K^*, \\ \Psi_0(v) \sup_{v \neq 0} \frac{\langle \xi, v\rangle}{\Psi_0(v)} & \text{if } \xi \notin K^*. \end{cases}$$

Therefore, in the particular case $\Psi_0(v) = A\|v\|_1$, taking into account that

$$K^* = \overline{B}^\infty_A(0) := \{\xi \in \mathbb{R}^d \ : \ \|\xi\|_\infty \leq A\},$$

we retrieve the formula

$$\mathsf{p}(0, v, \xi) = \|v\|_1 (A \vee \|\xi\|_\infty) \tag{5.15}$$

(here and in what follows, we use the notation $a \vee b$ for $\max\{a, b\}$).

(2) **2-norm vanishing-viscosity:** In this case, we have

$$\mathsf{p}(0,v,\xi) = \Psi_0(v) + \|v\| \min_{\zeta\in K^*} \|\xi-\zeta\|_*, \tag{5.16}$$

where we have used the notation $\|\zeta\|_* := \sup_{v\neq 0} \frac{\langle\zeta,v\rangle}{\|v\|}$. Clearly, (5.15) is a particular case of (5.16).

The stochastic approximation. We now consider the dissipation potentials Ψ_n from (2.5), i.e.

$$\begin{gathered}\Psi_n(v) = \sum_{i=1}^d \psi_n(v_i) = \sum_{i=1}^d \frac{v_i}{n} \log\left(\frac{v_i+\sqrt{v_i^2+e^{-2nA}}}{e^{-nA}}\right) - \frac{1}{n}\sqrt{v_i^2+e^{-2nA}} + \frac{e^{-nA}}{n}, \\ \text{with } \Psi_n^*(\xi) = \sum_{i=1}^d \psi_n^*(\xi_i) = \sum_{i=1}^d \frac{e^{-nA}}{n}\left(\cosh(n\xi_i)-1\right).\end{gathered} \tag{5.17}$$

Preliminarily, we observe that

$$\begin{cases}\Psi_n(v) \to \Psi_0(v) = A\|v\|_1 \quad \text{for all } v\in\mathbb{R}^d, \text{ and } \Gamma\text{-}\lim_{n\to\infty}\Psi_n = \Psi_0, \\ \Psi_n^*(\xi) \to I_{K^*}(\xi) \text{ with } K^* = \overline{B}_A^\infty(0) \quad \text{for all } \xi\in\mathbb{R}^d, \text{ and } \Gamma\text{-}\lim_{n\to\infty}\Psi_n^* = \Psi_0^*.\end{cases} \tag{5.18}$$

In order to check the above statement, e.g. for $\Psi_n(v)$, it is sufficient to recall that $\Psi_n(v) = \sum_{i=1}^d \psi_n(v_i)$, and that the real functions $(\psi_n)_n$ pointwise and Γ-converge to the 1-positively homogeneous potential $\psi_0 : \mathbb{R}\to\mathbb{R}$ given by $\psi_0(v) = A|v|$. We will now prove that the counterpart to Proposition 5.5 holds.

Proposition 5.7. *The dissipation potentials from* (5.17) *comply with* (4.6) *and with Hypothesis 4.1, with limiting viscosity contact potential*

$$\mathsf{p} : [0,+\infty)\times\mathbb{R}^d\times\mathbb{R}^d \to [0,+\infty] \text{ given by } \mathsf{p}(\tau,v,\xi) := \begin{cases}\Psi_0(v)+I_{K^*}(\xi) & \text{if } \tau>0, \\ \|v\|_1(A\vee\|\xi\|_\infty) & \text{if } \tau=0.\end{cases} \tag{5.19}$$

Proof. We will split the proof in several claims.

Claim 1: (5.19) **holds for** $\tau>0$**.** It follows from the Γ-convergence properties in (5.18) that $\mathsf{p} = \Gamma\text{-}\liminf_{n\to\infty}\mathsf{b}_{\Psi_n}$ fulfills $\mathsf{p}(\tau,v,\xi)\geq\Psi_0(v)+I_{K^*}(\xi)$ for all $(v,\xi)\in\mathbb{R}^d\times\mathbb{R}^d$, if $\tau>0$. For the converse inequality, for every $\xi\in\mathbb{R}^d$ we take the constant recovery sequence $\xi_n\equiv\xi$ and again choose for fixed $(\tau,v)\in[0,+\infty)\times\mathbb{R}^d$ the sequences $\tau_n\equiv\tau$ and $v_n\equiv v$. The pointwise convergences from (5.18) ensure that

$$\mathsf{p}(\tau,v,\xi) \leq \limsup_{n\to\infty}\mathsf{b}_{\Psi_n}(\tau,v,\xi) = \tau\Psi_0\left(\frac{v}{\tau}\right)+\tau I_{K^*}(\xi) = \Psi_0(v)+I_{K^*}(\xi).$$

Hence we conclude that $\mathsf{p}(\tau,v,\xi)=\Psi_0(v)+I_{K^*}(\xi)$, i.e. (5.14) for $\tau>0$.

Claim 2: (5.19) **holds for** $\tau=0$ **and** $v=0$**.** In this case we have to check that $\mathsf{p}(0,0,\xi)=0$, which is equivalent to showing that $\mathsf{p}(0,0,\xi)\leq 0$ as the functional p is positive. To this aim, for every fixed $\xi\in\mathbb{R}^d$ we observe that for any null sequence $\tau_n\downarrow 0$

$$\mathsf{p}(0,0,\xi) \leq \limsup_{n\to\infty}\mathsf{b}_{\Psi_n}(\tau_n,0,\xi) = \limsup_{n\to\infty}\tau_n\Psi_n^*(\xi),$$

and then we choose $(\tau_n)_n$ vanishing fast enough in such a way that the lim sup on the right-hand side equals zero.

Claim 3: (5.19) **holds for** $\tau=0$ **and** $v\neq 0$**.** We will split the proof in several (sub-)claims. In the following calculations, taking into account that $\Psi_n=\sum_{i=1}^d\psi_n$ and $\Psi_n^*=\sum_{i=1}^d\psi_n^*$ with $\psi_n:\mathbb{R}\to\mathbb{R}$ and $\psi_n^*:\mathbb{R}\to\mathbb{R}$ even functions, we will often confine the discussion to the case in which $v=(v_1,\ldots,v_d)$ fulfills $v_i\geq 0$ for all $i=1,\ldots,d$, and analogously for $\xi=(\xi_1,\ldots,\xi_d)$.

Moreover, we will need to work with the modified bipotentials $\mathsf{b}^{\delta}_{\Psi_n} : [0,+\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0,+\infty]$ given by

$$\mathsf{b}^{\delta}_{\Psi_n}(\tau, v, \xi) := \begin{cases} \tau \Psi_n\left(\frac{v}{\tau}\right) + \tau \Psi_n^*(\xi) + \tau\delta & \text{for } \tau > 0, \\ 0 & \text{for } \tau = 0,\ v = 0, \\ +\infty & \text{for } \tau = 0 \text{ and } v \neq 0 \end{cases} \tag{5.20}$$

with $\delta > 0$ <u>fixed</u>. We remark that $\operatorname{Argmin}_{\tau>0} \mathsf{b}^{\delta}_{\Psi_n}(\tau, v, \xi) \neq \emptyset$. Since for every $\xi \in \mathbb{R}^d$ the map $v \mapsto \min_{\tau>0} \mathsf{b}^{\delta}_{\Psi_n}(\tau, v, \xi)$ is in turn 1-positively homogeneous, there exists a closed convex set $K^*_{n,\delta}(\xi)$ such that

$$\min_{\tau>0} \mathsf{b}^{\delta}_{\Psi_n}(\tau, v, \xi) = \sup\left\{\langle v, w\rangle \ : \ w \in K^*_{n,\delta}(\xi)\right\}. \tag{5.21a}$$

Indeed, it turns out (cf. [MRS12a, Thm. A.17]) that

$$K^*_{n,\delta}(\xi) = \{w \in \mathbb{R}^d \ : \ \Psi_n^*(w) \le \Psi_n^*(\xi) + \delta\}. \tag{5.21b}$$

We need an intermediate estimate before proving the $\ge$-inequality in (5.19), i.e. (5.23) below.
Claim 3.1**:** there holds

$$\mathsf{p}(0, v, \xi) \ge \inf\{\liminf_{n\to\infty} \mathsf{b}^{\delta}_{\Psi_n}(\bar\tau^{\delta}_n, v_n, \xi_n) \ : v_n \to v,\ \xi_n \to \xi\}, \qquad \text{where } \bar\tau^{\delta}_n \in \operatorname*{Argmin}_{\tau} \mathsf{b}^{\delta}_{\Psi_n}(\tau, v_n, \xi_n)\,. \tag{5.22}$$

This follows from

$$\begin{aligned} \mathsf{p}(0, v, \xi) &= \inf\{\liminf_{n\to\infty} \mathsf{b}_{\Psi_n}(\tau_n, v_n, \xi_n) \ : \tau_n \to 0,\ v_n \to v,\ \xi_n \to \xi\} \\ &= \inf\{\liminf_{n\to\infty} \left(\mathsf{b}^{\delta}_{\Psi_n}(\tau_n, v_n, \xi_n)\right) - \delta\tau_n \ : \tau_n \to 0,\ v_n \to v,\ \xi_n \to \xi\} \\ &\overset{(2)}{\ge} \inf\{\liminf_{n\to\infty} \min_{\tau>0} \mathsf{b}^{\delta}_{\Psi_n}(\tau, v_n, \xi_n) \ : v_n \to v,\ \xi_n \to \xi\}\,, \end{aligned}$$

where (2) follows from the fact that $\lim_{n\to\infty} \delta\tau_n = 0$ for every vanishing sequence (τ_n).

Claim 3.2**:** there holds

$$\mathsf{p}(0, v, \xi) \ge \|v\|_1 (A \vee \|\xi\|_\infty). \tag{5.23}$$

In view of (5.22), it is sufficient to prove that

$$\inf\{\liminf_{n\to\infty} \mathsf{b}^{\delta}_{\Psi_n}(\bar\tau^{\delta}_n, v_n, \xi_n) \ : v_n \to v,\ \xi_n \to \xi\} \ge \|v\|_1 (A \vee \|\xi\|_\infty). \tag{5.24}$$

Hence, we fix a sequence $(v_n, \xi_n) \to (v, \xi)$ and, for n sufficiently big such that $\frac{1}{n}\log d < A$, define $w_n \in \mathbb{R}^d$ by

$$w_n := \left((A \vee \|\xi_n\|_\infty) - \frac{1}{n}\log d\,,\ \cdots\,,\ (A \vee \|\xi_n\|_\infty) - \frac{1}{n}\log d\right).$$

Taking into account the form (2.6) of Ψ_n^*, we estimate

$$\Psi_n^*(w_n) = d\frac{e^{-nA}}{n}\left(\cosh(n\|w_n\|_\infty) - 1\right)$$

distinguishing the two cases $\|\xi_n\|_\infty \le A$ and $\|\xi_n\|_\infty > A$. In the former situation, it is sufficient to observe that $\|w_n\|_\infty \le A$, so that

$$\Psi_n^*(w_n) \le d\frac{e^{-nA}}{n}\left(\cosh(nA) - 1\right) = \frac{d}{n}\left(\frac{1 + e^{-2nA} - 2e^{-nA}}{2}\right) \le \frac{d}{n} \le \delta \tag{5.25a}$$

for n sufficiently big. In the case $\|\xi_n\|_\infty > A$, we use that

$$
\begin{aligned}
\Psi_n^*(w_n) &= d\frac{e^{-nA}}{n}\left(\cosh(n\|\xi_n\|_\infty - \log(d)) - 1\right) \\
&= d\frac{e^{-nA}}{2n}\left(e^{n\|\xi_n\|_\infty - \log d} + e^{-n\|\xi_n\|_\infty + \log d} - 2\right) \\
&= \frac{e^{-nA}}{2n}\left(e^{n\|\xi_n\|_\infty} + d^2 e^{-n\|\xi_n\|_\infty} - 2d\right) \\
&= \frac{e^{-nA}}{2n}\left(e^{n\|\xi_n\|_\infty} + e^{-n\|\xi_n\|_\infty} - 2d\right) + \frac{e^{-nA}}{2n}(d^2-1)e^{-n\|\xi_n\|_\infty} \leq \Psi_n^*(\xi_n) + \delta
\end{aligned}
\tag{5.25b}
$$

for n sufficiently big such that $\frac{d^2-1}{2n} \leq \delta$. All in all, (5.25) gives that

$$\Psi_n^*(w_n) \leq \Psi_n^*(\xi_n) + \delta,$$

which implies that $w_n \in K_{n,\delta}(\xi_n)$, for all n sufficiently big. Now, using the representation formula (5.21a) for $\mathsf{b}_{\Psi_n}^\delta(\bar{\tau}_n^\delta, \cdot, \cdot)$, we find

$$\mathsf{b}_{\Psi_n}^\delta(\bar{\tau}_n^\delta, v_n, \xi_n) \geq \langle v_n, w_n \rangle = \|v_n\|_1 (A \vee \|\xi_n\|_\infty) - \frac{1}{n}\log d \|v_n\|_1,$$

where the last equality follows from the fact that $v_n = (v_n^1, \ldots, v_n^d)$ fulfills $v_n^i \geq 0$ for all $i = 1, \ldots, d$. Hence $\liminf_{n\to\infty} \mathsf{b}_{\Psi_n}^\delta(\bar{\tau}_n^\delta, v_n, \xi_n) \geq \|v\|_1 (A \vee \|\xi\|_\infty)$ and, since the sequences $(v_n)_n$ and $(\xi_n)_n$ are arbitrary, we conclude (5.24), and thus (5.23).

In order to prove the converse of inequality (5.23), and conclude (5.19), we preliminarily need to investigate the properties of the sets $K_{n,\delta}^*$.

Claim 3.3**:** there holds

$$\forall\, \delta > 0 \quad \exists\, n_\delta \in \mathbb{N}\ \forall\, n \geq n_\delta\ \forall\, w \in K_{n,\delta}^*(\xi): \quad \|w\|_\infty \leq A \vee \|\xi\|_\infty + \frac{1}{n}\log(2en\delta)\,. \tag{5.26}$$

Indeed, every $w \in K_{n,\delta}^*(\xi)$ fulfills $\Psi_n^*(w) \leq \Psi_n^*(\xi) + \delta$. Using the explicit formula for Ψ_n^* we obtain that

$$\frac{e^{-nA}}{n}\cosh(n\|w\|_\infty) \leq \frac{de^{-nA}}{n}\cosh(n\|\xi\|_\infty) + \delta,$$

whereby

$$\frac{e^{-nA}}{2n}e^{n\|w\|_\infty} \leq \frac{de^{-nA}}{2n}e^{n\|\xi\|_\infty} + \frac{de^{-nA}}{2n} + \delta \leq \frac{de^{-nA}}{n}e^{n\|\xi\|_\infty} + \delta,$$

and thus

$$\|w\|_\infty \leq \frac{1}{n}\log\left(2n\delta e^{nA} + 2de^{n\|\xi\|_\infty}\right).$$

Now, doing some algebraic manipulation on the logarithmic term on the right-hand side we find

$$
\begin{aligned}
\log\left(2n\delta e^{nA} + 2de^{n\|\xi\|_\infty}\right) &= \log\left(e^{nA + \log n\delta}\left(1 + e^{n(\|\xi\|_\infty - A) + \log d - \log n\delta}\right)\right) + \log 2 \\
&\stackrel{(1)}{\leq} \log\left(1 + e^{n(\|\xi\|_\infty - A)_+}\right) + nA + \log n\delta + \log 2 \\
&\stackrel{(2)}{\leq} n(A \vee \|\xi\|_\infty) + 1 + \log 2n\delta,
\end{aligned}
$$

where for (1) we have used that $n\delta > d$ for n sufficiently big and for (2) we have estimated $\log(1+e^{n(\|\xi\|_\infty - A)_+}) = \log(e^{n(\|\xi\|_\infty - A)_+}) + \log(e^{-n(\|\xi\|_\infty - A)_+} + 1) \leq \log(e^{n(\|\xi\|_\infty - A)_+}) + 1$. Then, (5.26) ensues.

Claim 3.4**:** for every $(v, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$ and all sequences $(v_n)_n$, $(\xi_n)_n$ with $v_n \to v$ and $\xi_n \to \xi$, for every $\bar{\tau}_n^\delta \in \operatorname{Argmin} \mathsf{b}_{\Psi_n}^\delta(\cdot, v_n, \xi_n)$ there holds

$$\lim_{n\to\infty} \bar{\tau}_n^\delta = 0. \tag{5.27}$$

We distinguish two cases: (1) $\Psi_0^*(\xi) = +\infty$ and (2) $\Psi_0^*(\xi) = 0$.

(1) In the first case, we have $\liminf_{n\to\infty} \Psi_n^*(\xi_n) = +\infty$. Then $\bar{\tau}_n^\delta$ must be vanishing to "cancel" the $\tau\Psi_n^*$-contribution, cf. also the proof of Lemma 4.6.

(2) In the second case, to show (5.27) we will provide an estimate from above for $\bar{\tau}_n^\delta$ by exploiting the Euler-Lagrange equation for the minimum problem $\min_{\tau>0} \mathsf{b}_{\Psi_n}^\delta(\tau, v, \xi)$. Namely, $\bar{\tau}_n^\delta$ complies with

$$0 \in \partial_\tau \mathsf{b}_{\Psi_n}^\delta(\cdot, v_n, \xi_n)(\bar{\tau}_n^\delta) = \Psi_n\left(\frac{v_n}{\bar{\tau}_n^\delta}\right) - \left\langle \partial\Psi_n\left(\frac{v_n}{\bar{\tau}_n^\delta}\right), \frac{v_n}{\bar{\tau}_n^\delta}\right\rangle + \Psi_n^*(\xi_n) + \delta \tag{5.28}$$

(where, with a slight abuse of notation, we have written $\partial\Psi_n(v_n/\bar{\tau}_n^\delta)$ as a singleton). Using the explicit formula (5.17) for Ψ_n we find

$$\Psi_n\left(\frac{v_n}{\bar{\tau}_n^\delta}\right) - \left\langle \partial\Psi_n\left(\frac{v_n}{\bar{\tau}_n^\delta}\right), \frac{v_n}{\bar{\tau}_n^\delta}\right\rangle = \frac{de^{-nA}}{n} - \sum_i \frac{1}{n}\sqrt{\frac{(v_n^i)^2}{(\bar{\tau}_n^\delta)^2} + e^{-2nA}}\,.$$

Therefore, (5.28) yields

$$n\delta + de^{-nA} + n\Psi_n^*(\xi_n) = \sum_i \sqrt{\frac{(v_n^i)^2}{(\bar{\tau}_n^\delta)^2} + e^{-2nA}} \le d\sqrt{\frac{\|v_n\|_\infty^2}{(\bar{\tau}_n^\delta)^2} + e^{-2nA}},$$

$$\text{whence} \quad (\bar{\tau}_n^\delta)^2 \le \frac{d^2\|v_n\|_\infty^2}{n^2\delta^2 + n^2\left(\Psi_n^*(\xi_n)\right)^2} \to 0 \quad \text{for } n \to \infty. \tag{5.29}$$

We are now in a position to conclude the proof of (5.19).
Claim 3.5**:** there holds

$$\mathsf{p}(0, v, \xi) \le \|v\|_1(A \vee \|\xi\|_\infty). \tag{5.30}$$

We will in fact prove that

$$\forall\, \xi \in \mathbb{R}^d \ \ \exists\, (\xi_n)_n \subset \mathbb{R}^d \ : \ \ \forall\, v \in \mathbb{R}^d\ \exists\, (\tau_n, v_n)_n \ \text{ s.t. } \begin{cases} \tau_n \to 0, \\ v_n \to v, \\ \limsup_{n\to\infty} \mathsf{b}_{\Psi_n}(\tau_n, v_n, \xi_n) \le \|v\|_1(A \vee \|\xi\|_\infty). \end{cases} \tag{5.31}$$

Taking into account that $\mathsf{p} = \Gamma\text{-}\liminf_{n\to\infty} \mathsf{b}_{\Psi_n}$, we will then conclude (5.30). To check (5.31), let us choose the constant recovery sequences $\xi_n \equiv \xi$ and $v_n \equiv v$, and let $\tau_n := \bar{\tau}_n^\delta \in \mathrm{Argmin}_{\tau>0} \mathsf{b}_{\Psi_n}^\delta(\tau, v, \xi)$. By the previous Claim 3.4, we have that $\tau_n \downarrow 0$. Now, in view of the representation formula (5.21a) for $\min_{\tau>0} \mathsf{b}_{\Psi_n}^\delta(\tau, v, \xi)$, we can construct a sequence $\{\tilde{\xi}_n\} \subset K_{n,\delta}^*(\xi)$ such that

$$\mathsf{b}_{\Psi_n}^\delta(\bar{\tau}_n^\delta, v, \xi) \le \langle v, \tilde{\xi}_n\rangle + \frac{1}{n} \le \|v\|_1(A \vee \|\xi\|_\infty) + \frac{\|v\|_1}{n}\log(2en\delta) + \frac{1}{n},$$

where the second estimate ensues from (5.26). Therefore $\limsup_{n\to\infty} \mathsf{b}_{\Psi_n}^\delta(\bar{\tau}_n^\delta, v, \xi) \le \|v\|_1(A \vee \|\xi\|_\infty)$. Since $\limsup_{n\to\infty} \mathsf{b}_{\Psi_n}^\delta(\bar{\tau}_n^\delta, v, \xi) = \limsup_{n\to\infty} \mathsf{b}_{\Psi_n}(\bar{\tau}_n^\delta, v, \xi)$ as the sequence $(\bar{\tau}_n^\delta)_n$ is vanishing, we conclude (5.31).

This finishes the proof of Proposition 5.7. □

5.3. **The Γ-limsup result.** For the Γ-lim sup counterpart to Theorem 5.2, where we now consider the *strict* topology in $\mathrm{BV}([0,T];\mathbb{R}^d)$, we will focus on the 1-positively homogeneous potential

$$\Psi_0(v) = A\|v\|_1 \quad \text{with } A > 0,$$

and the two following *specific cases*:

vanishing viscosity: the dissipation potentials Ψ_n are obtained by augmenting Ψ_0 by a quadratic term involving a (possibly) different norm $\|\cdot\|$ (cf. 5.12), i.e.

$$\Psi_n(v) = A\|v\|_1 + \frac{\varepsilon_n}{2}\|v\|^2 \text{ with } \varepsilon_n \downarrow 0,$$

$$\text{with limiting viscosity contact potential } \mathsf{p}(\tau, v, \xi) = \begin{cases} \Psi_0(v) + I_{K^*}(\xi) & \text{if } \tau > 0, \\ \Psi_0(v) + \|v\| \min_{\zeta \in K^*} \|\xi - \zeta\|_* & \text{if } \tau = 0; \end{cases} \tag{5.32}$$

stochastic approximation: the dissipation potentials Ψ_n are given by (5.17), with viscosity contact potential

$$\mathsf{p}(\tau, v, \xi) = \begin{cases} \Psi_0(v) + I_{K^*}(\xi) & \text{if } \tau > 0, \\ \|v\|_1 (A \vee \|\xi\|_\infty) & \text{if } \tau = 0. \end{cases}$$

In [BP16], which focused on *one-dimensional* rate-independent systems, the Γ-lim sup result was obtained in a much larger generality, for a class of dissipation potentials Ψ_n fulfilling suitable growth conditions and other properties. Such properties are satisfied in the two abovementioned particular cases.

We believe that, to some extent, the results in [BP16] could be extended to the present multi-dimensional context. Still, we have preferred to confine the discussion to the vanishing-viscosity and the stochastic approximations, in order to develop more explicit calculations than those in the proof of [BP16, Thm. 4.2], significantly exploiting the specific structure of these examples.

Finally, let us mention in advance that, like in [BP16], we will need to impose some enhanced regularity for $\mathcal{E}(t,\cdot)$, namely

$$\begin{aligned} &\exists\, C_{\mathsf{E}} > 0 \ \ \forall\, (t,u) \in [0,T] \times \mathbb{R}^d : \|\mathrm{D}\mathcal{E}(t,u)\| \leq C_{\mathsf{E}} \quad \text{and } \mathrm{D}\mathcal{E}(\cdot,u) \text{ is uniformly Lipschitz continuous, i.e.} \\ &\qquad \exists\, L_{\mathsf{E}} > 0 \ \forall\, t_1,\, t_2 \in [0,T] \ \forall\, u \in \mathbb{R}^d : \quad \|\mathrm{D}\mathcal{E}(t_1,u) - \mathrm{D}\mathcal{E}(t_2,u)\| \leq L_{\mathsf{E}} |t_1 - t_2|\,. \end{aligned} \tag{5.33}$$

Theorem 5.8. *Let $\mathcal{E}$ comply with* (E) *and with* (5.33)*, and let the dissipation potentials $(\Psi_n)_n$ be given either by* (5.17)*, or by* (5.32)*.*

Then, for every $u \in \mathrm{BV}([0,T];\mathbb{R}^d)$ there exists a sequence $(u_n)_n \subset \mathrm{AC}([0,T];\mathbb{R}^d)$, converging to u in the strict *topology of* $\mathrm{BV}([0,T];\mathbb{R}^d)$*, such that*

$$\limsup_{n\to\infty} \mathscr{J}_{\Psi_n,\mathcal{E}}(u_n) \leq \mathscr{J}_{\Psi_0,\mathsf{p},\mathcal{E}}(u). \tag{5.34}$$

Clearly, Theorems 5.2 and 5.8 (whose proof is also postponed to Section 6), yield the *Mosco-convergence* of the functionals $(\mathscr{J}_{\Psi_n,\mathcal{E}})_n$ to $\mathscr{J}_{\Psi_0,\mathsf{p},\mathcal{E}}$, with respect to the weak-strict topology of $\mathrm{BV}([0,T];\mathbb{R}^d)$, in the vanishing-viscosity and stochastic cases.

Another straightforward consequence of Theorem 5.8, in the spirit of Corollary 5.3, is the following *reverse approximation* result.

Corollary 5.9. *Let $\mathcal{E}$ comply with* (E) *and with* (5.33)*. Let $\Psi_0(v) = A\|v\|_1$ and $\mathsf{p}(0,v,\xi) = \|v\|_1(A \vee \|\xi\|_\infty)$.*

Then, for every Balanced Viscosity solution $u \in \mathrm{BV}([0,T];\mathbb{R}^d)$ to the rate-independent system $(\Psi_0,\mathsf{p},\mathcal{E})$ there exists a sequence $(u_n)_n \subset \mathrm{AC}([0,T];\mathbb{R}^d)$ of solutions to the gradient systems $(\Psi_n,\mathcal{E})$, with the dissipation potentials $(\Psi_n)_n$ given by $\Psi_n(v) = A\|v\|_1 + \frac{\varepsilon_n}{2}\|v\|_1^2$ for all $n \in \mathbb{N}$, with $\varepsilon_n \downarrow 0$ as $n \to \infty$ (Ψ_0-vanishing viscosity*), such that $u_n \to u$ as $n \to \infty$* strictly *in* $\mathrm{BV}([0,T];\mathbb{R}^d)$.

A completely analogous statement holds with the dissipation potentials $(\Psi_n)_n$ from (5.17)*.*

6. Proofs of Theorems 5.2 and 5.8

In what follows, we will denote by C a generic positive constant independent of n, whose meaning may vary even within the same line.

We will just outline the argument for the proof of **Proposition 5.1**, referring to the argument for [MRS12b, Thm. 4.1] (see also [BP16, Thm. 4.2]) for all details. Combining the information that $\mathscr{J}_{\Psi_n,\mathcal{E}}(u_n) \leq C$ with the power control condition from (E), we find that

$$\int_0^T \left(\Psi_n(\dot{u}(s)) + \Psi_n^*(-\mathrm{D}\mathcal{E}(t,u(s)))\right) \mathrm{d}s + \mathcal{E}(T,u_n(T)) \leq C + \int_0^T C_1 |\mathcal{E}(s,u_n(s))| \,\mathrm{d}s,$$

where we have also used that $\|u_n(0)\| \leq C$, and thus $\sup_n |\mathcal{E}(0,u_n(0))| \leq C$. Taking into account that both Ψ_n and Ψ_n^* are positive, via the Gronwall Lemma we deduce from the above inequality that $\sup_{t\in[0,T]} |\mathcal{E}(t,u_n(t))| \leq C$, whence $\sup_{t\in[0,T]} |\partial_t \mathcal{E}(t,u_n(t))| \leq C$. Hence

$$\int_0^T \left(\Psi_n(\dot{u}_n(s)) + \Psi_n^*(-\mathrm{D}\mathcal{E}(t,u(s)))\right) \mathrm{d}s \leq C,$$

which implies thanks to the coercivity (5.4) that $\mathrm{Var}(u_n;[0,T]) \le C$. Then, the thesis readily follows from the Helly theorem. ■

Before developing the proof of Theorem 5.2, we preliminarily give the following lower semicontinuity result, in the spirit of [MRS09, Lemma 3.1], cf. also [MRS12b, Lemma 4.3]).

Lemma 6.1. *Let $m,\, d \ge 1$ and $\mathfrak{F}_n,\, \mathfrak{F}_\infty : \mathbb{R}^m \times \mathbb{R}^d \to [0,+\infty)$ be normal integrands such that*

(1) for fixed $\xi \in \mathbb{R}^d$ the functionals $\mathfrak{F}_n(\cdot,\xi)$ are convex for every $n \in \mathbb{N} \cup \{\infty\}$,
(2) there holds

$$\Gamma\text{-}\liminf_{n\to\infty} \mathfrak{F}_n \ge \mathfrak{F}_\infty \quad \text{in } \mathbb{R}^m \times \mathbb{R}^d. \tag{6.1}$$

Let I be a bounded interval in $\mathbb{R}$ and let $w_n,\, w : I \to \mathbb{R}^m$ fulfill $w_n \rightharpoonup w$ in $L^1(I;\mathbb{R}^m)$, and $\xi_n,\, \xi : I \to \mathbb{R}^d$ fulfill $\xi_n(s) \to \xi(s)$ for almost all $s \in I$. Then

$$\liminf_{n\to\infty} \int_I \mathfrak{F}_n(w_n(s),\xi_n(s))\,\mathrm{d}s \ge \int_I \mathfrak{F}_\infty(w(s),\xi(s))\,\mathrm{d}s. \tag{6.2}$$

Proof. We introduce the functional

$$\overline{\mathfrak{F}} : \mathbb{N}\cup\{\infty\} \times \mathbb{R}^m \times \mathbb{R}^d, \qquad \overline{\mathfrak{F}}(n,w,\xi) := \begin{cases} \mathfrak{F}_n(w,\xi) & \text{for } n \in \mathbb{N}, \\ \mathfrak{F}_\infty(w,\xi) & \text{for } n = \infty. \end{cases}$$

It follows from (6.1) that $\overline{\mathfrak{F}}$ is lower semicontinuous on $\mathbb{N}\cup\{\infty\} \times \mathbb{R}^m \times \mathbb{R}^d$, hence it is a positive normal integrand. Then, (6.2) follows from the Ioffe Theorem, cf. [Iof77] and also, e.g., [Val90, Thm. 21]. □

Proof of Theorem 5.2. Let $(u_n)_n \subset \mathrm{BV}([0,T];\mathbb{R}^d)$ be a sequence weakly converging to $u \in \mathrm{BV}([0,T],\mathbb{R}^d)$. We may suppose that $\liminf_{n\to\infty} \mathscr{J}_{\Psi_n,\mathcal{E}}(u_n) < +\infty$, as otherwise there is nothing to prove. Therefore, up to a subsequence we have $\mathscr{J}_{\Psi_n,\mathcal{E}}(u_n) \le C$, in particular yielding that $u_n \in \mathrm{AC}([0,T];\mathbb{R}^d)$ for every $n \in \mathbb{N}$. With the very same arguments as in the proof of Prop. 5.1, also based on the power control (E), we see that each contribution to $\mathscr{J}_{\Psi_n,\mathcal{E}}(u_n)$ is itself bounded. Convergences (5.6) follow from the pointwise convergence of $(u_n)_n$, the fact that $\mathcal{E} \in \mathrm{C}^1([0,T]\times\mathbb{R}^d)$, and the Lebesgue dominated convergence theorem, recalling that $(u_n)_n$ is bounded in $L^\infty(0,T;\mathbb{R}^d)$. Moreover, we have that $\mathrm{D}\mathcal{E}(t,u_n(t)) \to \mathrm{D}\mathcal{E}(t,u(t))$ for every $t \in [0,T]$. Then, taking into account that the functionals $(\Psi_n^*)_n$ Γ-converge to Ψ_0^*, we can apply Lemma 6.1 to the functionals $\mathfrak{F}_n(w,\xi) := \Psi_n^*(\xi)$ and $\mathfrak{F}(w,\xi) := \Psi_0^*(w)$ to obtain

$$\begin{gathered} \liminf_{n\to\infty} \int_0^T \Psi_n^*(-\mathrm{D}\mathcal{E}(t,u_n(t)))\,\mathrm{d}t \ge \int_0^T \Psi_0^*(-\mathrm{D}\mathcal{E}(t,u(t)))\,\mathrm{d}t, \\ \text{whence} \qquad -\mathrm{D}\mathcal{E}(t,u(t)) \in K^* \text{ for a.a. } t \in (0,T). \end{gathered} \tag{6.3}$$

Define the non-negative finite measures on $[0,T]$

$$\nu_n := \Psi_n(\dot{u}_n(\cdot))\mathscr{L}^1 + \Psi_n^*(-\mathrm{D}\mathcal{E}(\cdot,u_n(\cdot)))\mathscr{L}^1 \doteq \mu_n + \eta_n.$$

Up to extracting a subsequence, we can suppose that they weakly* converge to a positive measure

$$\nu = \mu + \eta \qquad \text{with } \eta \ge \Psi_0^*(-\mathrm{D}\mathcal{E}(\cdot,u(\cdot)))\mathscr{L}^1.$$

Let us now preliminarily show that

$$\nu \ge \Psi_0(\dot{u})\mathscr{L}^1 + \mu_{\Psi_0,\mathrm{C}}. \tag{6.4}$$

For this, we shall in fact observe that $\mu \ge \Psi_0(\dot{u})\mathscr{L}^1 + \mu_{\Psi_0,\mathrm{C}}$. This will follow upon proving that

$$\mu([\alpha,\beta]) = \lim_{n\to\infty} \int_\alpha^\beta \Psi_n(\dot{u}_n(t))\,\mathrm{d}t \ge \mathrm{Var}_{\Psi_0}(u;[\alpha,\beta]) \quad \text{for every } [\alpha,\beta] \subset [0,T]. \tag{6.5}$$

Indeed, let us fix a partition $t_0 = \alpha < t_1 < \ldots < t_k = \beta$ of $[\alpha, \beta]$ and notice that

$$
\begin{aligned}
\lim_{n\to\infty} \int_\alpha^\beta \Psi_n(\dot{u}_n(t))\,\mathrm{d}t = \lim_{n\to\infty} \sum_{m=1}^k \int_{t_{m-1}}^{t_m} \Psi_n(\dot{u}_n(t))\,\mathrm{d}t &\overset{(1)}{\geq} \liminf_{n\to\infty} \sum_{m=1}^k (t_m - t_{m-1}) \Psi_n \left(\frac{\int_{t_{m-1}}^{t_m} \dot{u}_n(t)\,\mathrm{d}t}{t_m - t_{m-1}} \right) \\
&= \liminf_{n\to\infty} \sum_{m=1}^k (t_m - t_{m-1}) \Psi_n \left(\frac{u_n(t_m) - u_n(t_{m-1})}{t_m - t_{m-1}} \right) \\
&\overset{(2)}{\geq} \sum_{m=1}^k (t_m - t_{m-1}) \Psi_0 \left(\frac{u(t_m) - u(t_{m-1})}{t_m - t_{m-1}} \right) \\
&\overset{(3)}{=} \sum_{m=1}^k \Psi_0 \left(u(t_m) - u(t_{m-1}) \right),
\end{aligned}
$$

where (1) follows from the Jensen inequality, (2) from the fact that the potentials $(\Psi_n)_n$ Γ-converge to Ψ_0 (cf. Lemma 4.4), and (3) from the 1-positive homogeneity of Ψ_0. Since the partition of $[\alpha, \beta]$ is arbitrary, we conclude (6.5).

However, we need to improve (6.4) by obtaining a finer characterization for the jump part of ν. We will in fact prove that

$$
\nu(\{t\}) \geq \Delta_{\mathsf{p},\mathcal{E}}(t; u(t_-), u(t_+)) \qquad \text{for every } t \in \mathrm{J}_u \tag{6.6}
$$

by adapting the argument in the proof of [MRS16, Prop. 7.3]. To this end, for fixed $t \in \mathrm{J}_u$ let us pick two sequences $h_n^- \uparrow t$ and $h_n^+ \downarrow t$ such that $u_n(h_n^-) \to u(t_-)$ and $u_n(h_n^+) \to u(t_+)$. Define $\mathsf{s}_n : [h_n^-, h_n^+] \to \mathbb{R}$ by

$$
\mathsf{s}_n(h) := c_n \left(h - h_n^- + \int_{h_n^-}^h \left(\Psi_n(\dot{u}_n(t)) + \Psi_n^*(-\mathrm{D}\mathcal{E}(t, u_n(t))) \right) \mathrm{d}t \right), \qquad h \in [h_n^-, h_n^+], \tag{6.7}
$$

where the normalization constant c_n is chosen in such a way that $\mathsf{s}_n(h_n^+) = 1$. Therefore, s_n takes values in $[0, 1]$. Observe that for every n the function s_n is strictly increasing and thus invertible, and let

$$
\mathsf{t}_n := \mathsf{s}_n^{-1} : [0,1] \to [h_n^-, h_n^+] \quad \text{and} \quad \vartheta_n := u_n \circ \mathsf{t}_n.
$$

There holds

$$
\dot{\mathsf{t}}_n(s) + \|\dot{\vartheta}_n\|_1(s) = \frac{1 + \|\dot{u}_n\|_1(\mathsf{t}_n(s))}{c_n \left(1 + \Psi_n(\dot{u}_n(\mathsf{t}_n(s))) + \Psi_n^*(-\mathrm{D}\mathcal{E}(\mathsf{t}_n(s), u_n(\mathsf{t}_n(s))))\right)} \leq C \quad \text{for a.a. } s \in (0,1). \tag{6.8}
$$

Now, by the upper semicontinuity property of the weak*-convergence of measures on closed sets we have

$$
\begin{aligned}
\nu(\{t\}) \geq \limsup_{n\to\infty} \nu_n([h_n^-, h_n^+]) &\geq \liminf_{n\to\infty} \int_{h_n^-}^{h_n^+} \left(\Psi_n(\dot{u}_n(t)) + \Psi_n^*(-\mathrm{D}\mathcal{E}(t, u_n(t))) \right) \mathrm{d}t \\
&\overset{(1)}{=} \liminf_{n\to\infty} \int_0^1 \left(\Psi_n(\dot{u}_n(\mathsf{t}_n(s))) + \Psi_n^*(-\mathrm{D}\mathcal{E}(\mathsf{t}_n(s), u_n(\mathsf{t}_n(s)))) \right) \dot{\mathsf{t}}_n(s)\,\mathrm{d}s \\
&\overset{(2)}{=} \liminf_{n\to\infty} \int_0^1 \mathsf{b}_{\Psi_n}(\dot{\mathsf{t}}_n(s), \dot{\vartheta}_n(s), -\mathrm{D}\mathcal{E}(\mathsf{t}_n(s), \vartheta_n(s)))\,\mathrm{d}s
\end{aligned} \tag{6.9}
$$

where (1) follows from a change of variables, and (2) from the very definition (4.1) of b_{Ψ_n}. Now, it follows from (6.8) and from the fact that the range of t_n is $[h_n^-, h_n^+]$ that there exists $(\mathsf{t}, \vartheta) \in \mathrm{C}^0_{\mathrm{lip}}([0,1]; [0,T] \times \mathbb{R}^d)$ such that, up to a not relabeled subsequence,

$$
\begin{gathered}
\mathsf{t}_n(s) \to \mathsf{t}(s) \equiv t, \quad \vartheta_n(s) \to \vartheta(s) \text{ for all } s \in [0,1], \qquad \dot{\mathsf{t}}_n \rightharpoonup^* 0 \text{ in } L^\infty(0,1), \quad \dot{\vartheta}_n \rightharpoonup^* \dot{\vartheta} \text{ in } L^\infty(0,1;\mathbb{R}^d), \\
\text{so that } \vartheta(0) = \lim_{n\to\infty} u_n(h_n^-) = u(t_-) \text{ and } \vartheta(1) = \lim_{n\to\infty} u_n(h_n^+) = u(t_+).
\end{gathered} \tag{6.10}
$$

Therefore, applying Lemma 6.1 above with the choices $m = d+1$ and, for $w = (\tau, v) \in \mathbb{R} \times \mathbb{R}^d$, with $\mathfrak{F}_n(w, \xi) = \mathfrak{F}_n(\tau, v, \xi) := \mathsf{b}_{\Psi_n}(\tau, v, \xi)$ and $\mathfrak{F}_\infty(w, \xi) := \mathsf{p}(\tau, v, \xi)$ (where we still denote by b_{Ψ_n} and by p their t extensions to $\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d$ by infinity), and taking into account (4.2) from Hyp. 4.1, which ensures the validity of condition (6.1) in Lemma 6.1, we conclude

$$\liminf_{n\to\infty}\int_0^1 \mathsf{b}_{\Psi_n}(\dot{\mathsf{t}}_n(s),\dot\vartheta_n(s),-\mathrm{D}\mathcal{E}(\mathsf{t}_n(s),\vartheta_n(s)))\,\mathrm{d}s \geq \int_0^1 \mathsf{p}(0,\dot\vartheta(s),-\mathrm{D}\mathcal{E}(t,\vartheta(s)))\,\mathrm{d}s \geq \Delta_{\mathsf{p},\mathcal{E}}(t;u(t_-),u(t_+))\,.$$

Similarly, we prove that

$$\limsup_{n\to\infty}\nu_n([h_n^-,t]) \geq \Delta_{\mathsf{p},\mathcal{E}}(t;u(t_-),u(t)), \qquad \limsup_{n\to\infty}\nu_n([t,h_n^+]) \geq \Delta_{\mathsf{p},\mathcal{E}}(t;u(t),u(t_+)).$$

Repeating the very same arguments as in the proof of [MRS16, Prop. 7.3], we ultimately find that

$$\liminf_{n\to\infty}\int_s^t \left(\Psi_n(\dot u_n(r))+\Psi_n^*(-\mathrm{D}\mathcal{E}(r,u_n(r)))\right)\mathrm{d}r \geq \mathrm{Var}_{\Psi_0,\mathsf{p},\mathcal{E}}(u;[s,t]) \quad \text{for every } 0\leq s\leq t,$$

whence (5.7) also in view of (6.3). This concludes the proof. ∎

Proof of Corollary 5.3. Let $u\in \mathrm{BV}([0,T];\mathbb{R}^d)$ be a limit point of the sequence $(u_n)_n\subset \mathrm{AC}([0,T];\mathbb{R}^d)$. It follows from (5.5) that $\mathscr{J}_{\Psi_0,\mathsf{p},\mathcal{E}}(u)=0$, hence by Prop. 3.6 u is a Balanced Viscosity solution to $(\Psi_0,\mathsf{p},\mathcal{E})$. Moreover, for every $0\leq s\leq t\leq T$ we have that

$$\begin{aligned}
&\limsup_{n\to\infty}\int_s^t \left(\Psi_n(\dot u_n(r))+\Psi_n^*(-\mathrm{D}\mathcal{E}(r,u_n(r)))\right)\mathrm{d}r\\
&\overset{(1)}{\leq} \limsup_{n\to\infty}\left(\mathcal{E}(s,u_n(s))-\mathcal{E}(t,u_n(t))+\int_s^t \partial_t\mathcal{E}(r,u_n(r))\,\mathrm{d}r+\varepsilon_n\right)\\
&\overset{(2)}{=} \mathcal{E}(s,u(s))-\mathcal{E}(t,u(t))+\int_s^t \partial_t\mathcal{E}(r,u(r))\,\mathrm{d}r\\
&\overset{(3)}{=} \mathrm{Var}_{\Psi_0,\mathsf{p},\mathcal{E}}(u;[s,t])+\int_s^t \Psi_0^*(-\mathrm{D}\mathcal{E}(t,u(r)))\,\mathrm{d}r
\end{aligned}$$

where (1) follows from $\mathscr{J}_{\Psi_n}(u_n)\leq\varepsilon_n$, (2) from convergences (5.6), and (3) from the fact that $\mathscr{J}_{\Psi_0,\mathsf{p},\mathcal{E}}(u)=0$. Combining this with (5.7), we conclude the enhanced convergences (5.8). ∎

<u>Proof of Theorem 5.8</u>. Given $u\in\mathrm{BV}([0,T],\mathbb{R}^d)$, we will construct a sequence $(u_n)_n\subset\mathrm{AC}([0,T];\mathbb{R}^d)$ such that $u_n\to u$ strictly in $\mathrm{BV}([0,T];\mathbb{R}^d)$ and

$$\limsup_{n\to\infty}\mathcal{J}_{\Psi_n,\mathcal{E}}(u_n)\leq \mathscr{J}_{\Psi_0,\mathsf{p},\mathcal{E}}(u). \tag{6.11}$$

We split the proof of in several steps; for Steps 1–4, we shall adapt the arguments from the proof of [BP16, Thm. 4.2].

Step 1*: reparametrisation.* First we reparametrise the curve u, in terms of a new time-like parameter s on a domain $[0,S]$. The aim is to expand the jumps in u into smooth connections. Following [MRS12a, Prop. 6.9], we define

$$\mathsf{s}(t):=t+\mathrm{Var}_{\Psi_0,\mathsf{p},\mathcal{E}}(u;[0,t]).$$

Then there exists a Lipschitz parametrisation $(\mathsf{t},\mathsf{u}):[0,S]\to[0,T]\times\mathbb{R}$ such that t is non-decreasing,

$$\mathsf{t}(\mathsf{s}(t))=t \qquad \text{and} \qquad \mathsf{u}(\mathsf{s}(t))=u(t) \text{ for every } t\in[0,T], \tag{6.12}$$

and such that

$$\int_0^S \mathsf{p}(\dot{\mathsf{t}}(s),\dot{\mathsf{u}}(s),-\mathrm{D}\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s))\,\mathrm{d}s = \mathrm{Var}_{\Psi_0,\mathsf{p},\mathcal{E}}(u;[0,T])+\int_0^T \Psi_0^*(-\mathrm{D}\mathcal{E}(t,u(t)))\,\mathrm{d}t\,. \tag{6.13}$$

Moreover, it also holds that

$$\mathrm{Var}_{\Psi_0}(\mathsf{u};[0,S])=\mathrm{Var}_{\Psi_0}(u;[0,T]). \tag{6.14}$$

Step 2: preliminary remarks. Since we will construct a sequence $(u_n)_n$ strictly (and in particular pointwise) converging to u in $\mathrm{BV}([0,T];\mathbb{R}^d)$, thanks to the smoothness of $\mathcal{E}$ (cf. (E)), we will have for the first three contributions to $\mathcal{J}_{\Psi_n,\mathcal{E}}(u_n)$

$$\mathcal{E}(T,u_n(T))-\mathcal{E}(0,u_n(0))-\int_0^T\partial_t\mathcal{E}(t,u_n(t))\,\mathrm{d}t\to\mathcal{E}(T,u(T))-\mathcal{E}(0,u(0))-\int_0^T\partial_t\mathcal{E}(t,u(t))\,\mathrm{d}t$$

as $n\to\infty$. Therefore, in order to prove (6.11) it will be sufficient to focus on the other terms in $\mathcal{J}_{\Psi_n,\mathcal{E}}$ and $\mathcal{J}_{\Psi_0,\mathsf{p},\mathcal{E}}$. In view of (6.13), it will be sufficient to prove that

$$\limsup_{n\to\infty}\int_0^T\left[\Psi_n\left(\dot u_n(t)\right)+\Psi_n^*\left(\mathrm{D}\mathcal{E}(t,u_n(t))\right)\right]\,\mathrm{d}t\le\int_0^S\mathsf{p}(\dot{\mathsf{t}}(s),\dot{\mathsf{u}}(s),\mathrm{D}\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s)))\,\mathrm{d}s. \tag{6.15}$$

Step 3: definition of the new time t_n and of the recovery sequence u_n. For the sake of simplicity, in what follows we construct a recovery sequence for a curve u *with jumps only at* 0 *and* T, postponing to the end of the proof (cf. *Step* 7), the discussion of the case of a curve with countably many jumps. We define u_n by first perturbing the time variable t: we fix $\delta>0$ and consider a selection

$$\tau_n^\delta(s)\in\mathop{\mathrm{Argmin}}_{\tau>0}\mathsf{b}^\delta_{\Psi_n}(\tau,\dot{\mathsf{u}}(s),-\mathrm{D}\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s))). \tag{6.16}$$

We define $\mathsf{t}_n:[0,S]\to[0,T_n]$ as the solution of the differential equation

$$\mathsf{t}_n(0)=0,\qquad\dot{\mathsf{t}}_n(s)=\dot{\mathsf{t}}(s)\vee\tau_n^\delta(s)\,. \tag{6.17}$$

Observe that $\dot{\mathsf{t}}(s)=0$ in $[0,\mathsf{s}(0^+)]\cup\in[\mathsf{s}(T^-),S]$, but we can assume that $|\dot{\mathsf{u}}(s)|>0$ on $[0,\mathsf{s}(0^+)]\cup\in[\mathsf{s}(T^-),S]$. This will be sufficient to guarantee that $\mathrm{Argmin}_{\tau>0}\,\mathsf{b}^\delta_{\Psi_n}(\tau,\dot{\mathsf{u}},\mathrm{D}\mathcal{E}(\mathsf{t},\mathsf{u}))\neq\emptyset$ on the latter set, and thus that τ_n^δ is well defined. On the other hand, for $s\in[\mathsf{s}(0^+),\mathsf{s}(T^-)]$, we have $\dot{\mathsf{t}}(s)=\left.\frac{1}{\dot{\mathsf{s}}(t)}\right|_{t=\mathsf{t}(s)}>0$. All in all, $\dot{\mathsf{t}}_n(s)>0$ for all $s\in[0,S]$. The range of t_n is $[0,T_n]$, with $T_n\ge T$; since the recovery sequence u_n has to be defined on the interval $[0,T]$, we rescale t_n by

$$\lambda_n:=\frac{T_n}{T}\ge 1, \tag{6.18}$$

and define our recovery sequence as follows:

$$u_n(t):=\mathsf{u}\left(\mathsf{t}_n^{-1}\left(t\lambda_n\right)\right),\qquad\text{so that}\qquad\dot u_n(t)=\frac{\dot{\mathsf{u}}}{\dot{\mathsf{t}}_n}\left(\mathsf{t}_n^{-1}\left(t\lambda_n\right)\right)\lambda_n. \tag{6.19}$$

Now we substitute the explicit formula for u_n, we perform a change of variable and obtain

$$\begin{aligned}&\int_0^T\Big[\Psi_n\left(\dot u_n(t)\right)+\Psi_n^*\left(t,-\mathrm{D}\mathcal{E}(t,u_n(t))\right)\Big]\,\mathrm{d}t\\&=\int_0^T\left[\Psi_n\left(\frac{\dot{\mathsf{u}}}{\dot{\mathsf{t}}_n}\left(\mathsf{t}_n^{-1}\left(t\lambda_n\right)\right)\lambda_n\right)+\Psi_n^*\left(-\mathrm{D}\mathcal{E}\big(t,\mathsf{u}(\mathsf{t}_n^{-1}(t\lambda_n))\big)\right)\right]\,\mathrm{d}t\\&=\int_0^S\left[\Psi_n\left(\frac{\dot{\mathsf{u}}}{\dot{\mathsf{t}}_n}(s)\lambda_n\right)+\Psi_n^*\Big(-\mathrm{D}\mathcal{E}(\mathsf{t}_n(s)\lambda_n^{-1},\mathsf{u}(s))\Big)\right]\frac{\dot{\mathsf{t}}_n(s)}{\lambda_n}\,\mathrm{d}s,\end{aligned}$$

so that

$$\int_0^T\Big[\Psi_n\left(\dot u_n(t)\right)+\Psi_n^*\left(-\mathrm{D}\mathcal{E}(t,u_n(t))\right)\Big]\,\mathrm{d}t=\int_0^S\mathsf{b}_{\Psi_n}(\lambda_n^{-1}\dot{\mathsf{t}}_n(s),\dot{\mathsf{u}}(s),-\mathrm{D}\mathcal{E}(\mathsf{t}_n(s)\lambda_n^{-1},\mathsf{u}(s)))\,\mathrm{d}s.$$

Step 4: Strict convergence of $(u_n)_n$. Recall that we need to prove the pointwise convergence $u_n(t)\to u(t)$ for all $t\in[0,T]$ and the convergence of the variations. For this, it will be crucial to have the following property, that shall be verified (even uniformly w.r.t. $s\in[0,S]$) both in the stochastic (cf. (6.24)), and in the vanishing-viscosity cases (cf. (6.38)):

$$\tau_n^\delta(s)\to 0\quad\text{as }n\to\infty\quad\text{for a.a. }s\in(0,S). \tag{6.20}$$

This implies that $\dot{\mathsf{t}}_n(s) \to \dot{\mathsf{t}}(s)$ for almost all $s \in (0,S)$, and then it will hold

$$\mathsf{t}_n(s) \to \mathsf{t}(s) \quad \text{for every } s \in [0,S] \qquad \Longrightarrow \qquad \lambda_n \to 1, \quad \mathsf{t}_n^{-1}(t\lambda_n) \to \mathsf{s}(t) \quad \text{for every } t \in [0,T].$$

Moreover, $\dot{\mathsf{t}}_n(s) > 0$ implies that $\mathsf{t}_n^{-1}(0) = 0$ and $\mathsf{t}_n^{-1}(T_n) = S$, and so we will have the desired pointwise convergence

$$u_n(t) = \mathsf{u}\left(\mathsf{t}_n^{-1}(t\lambda_n)\right) \to \mathsf{u}(\mathsf{s}(t)) \overset{(6.12)}{=} u(t) \qquad \text{for every } t \in [0,T].$$

The convergence of the variations will be automatic, since by definition of u_n we will have

$$\int_0^T A\|\dot{u}_n(t)\|_1 \,\mathrm{d}t = \int_0^S A\|\dot{\mathsf{u}}(s)\|_1 \,\mathrm{d}s = \mathrm{Var}_{\Psi_0}(\mathsf{u};[0,S]) \overset{(6.14)}{=} \mathrm{Var}_{\Psi_0}(u;[0,T]).$$

Therefore, from now on we will concentrate on the proof of the lim sup estimate (6.15).

Step 5: strategy for (6.15). First of all, we will show the following *pointwise* lim sup-inequality

$$\limsup_{n\to\infty} \mathsf{b}_{\Psi_n}(\lambda_n^{-1}\dot{\mathsf{t}}_n(s), \dot{\mathsf{u}}(s), -\mathrm{D}\mathcal{E}(\mathsf{t}_n(s)\lambda_n^{-1}, \mathsf{u}(s))) \le \mathsf{p}(\dot{\mathsf{t}}(s), \dot{\mathsf{u}}(s), -\mathrm{D}\mathcal{E}(\mathsf{u}(s), \mathsf{t}(s))) \quad \text{for a.a. } s \in (0,S)\,. \tag{6.21}$$

Secondly, we will apply the following version of the Fatou Lemma

$$\left.\begin{array}{rr} \limsup_{n\to\infty} f_n(s) \le f(s) & \text{for a.a. } s \in (0,S), \\ f_n(s) \le g_n(s) & \text{for a.a. } s \in (0,S), \\ g_n \to g & \text{in } L^1(0,S), \end{array}\right\} \Longrightarrow \limsup_{n\to\infty} \int_0^S f_n(s)\,\mathrm{d}s \le \int_0^S f(s)\,\mathrm{d}s, \tag{6.22}$$

for measurable functions $(f_n)_n$ and f, in order to conclude

$$\limsup_{n\to\infty} \int_0^S \mathsf{b}_{\Psi_n}(\lambda_n^{-1}\dot{\mathsf{t}}_n(s), \dot{\mathsf{u}}(s), -\mathrm{D}\mathcal{E}(\mathsf{t}_n(s)\lambda_n^{-1}, \mathsf{u}(s)))\,\mathrm{d}s \le \int_0^S \mathsf{p}(\dot{\mathsf{t}}(s), \dot{\mathsf{u}}(s), -\mathrm{D}\mathcal{E}(\mathsf{u}(s), \mathsf{t}(s)))\,\mathrm{d}s, \tag{6.23}$$

whence (6.15) and ultimately (6.11). For the proof of (6.21) and (6.23), we will distinguish the *stochastic* and the *vanishing-viscosity* cases.

Step 6a: proof of (6.21) *and* (6.23) *for* Ψ_n *given by* (5.17) *(stochastic approximation).* Preliminarily, we observe that, with the very same calculations as for (5.29) (cf. Claim 3.4 in the proof of Proposition 5.7), one has

$$\begin{aligned} &\tau_n^\delta(s) \le \sqrt{\frac{d^2\|\dot{u}(s)\|_\infty^2}{n^2\delta^2 + n^2\left(\Psi_n^*(-\mathrm{D}\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s)))\right)^2}} \to 0 \quad \text{for almost all } s \in (0,S), \quad \text{and thus} \\ &\sup_{s\in[0,S]} \tau_n^\delta(s) \le \frac{C}{\delta n} \to 0 \quad \text{as } n \to \infty, \end{aligned} \tag{6.24}$$

(with a slight abuse of notation, we use the symbol sup also for an essential supremum) where we have exploited the Lipschitz continuity of u. In order to prove the pointwise inequality (6.21), we start with the following algebraic manipulation

$$\begin{aligned} \mathsf{b}_{\Psi_n}(\lambda_n^{-1}\dot{\mathsf{t}}_n(s), \dot{\mathsf{u}}(s), -\mathrm{D}\mathcal{E}(\mathsf{t}_n(s)\lambda_n^{-1}, \mathsf{u}(s))) = {} & \mathsf{b}^\delta_{\Psi_n}(\tau_n^\delta(s), \dot{\mathsf{u}}(s), -\mathrm{D}\mathcal{E}(\mathsf{t}(s), \mathsf{t}(s))) - \tau_n^\delta(s)\delta \\ & + \dot{\mathsf{t}}_n(s)\Psi_n^*(-\mathrm{D}\mathcal{E}(\mathsf{t}_n(s)\lambda_n^{-1}, \mathsf{u}(s))) - \tau_n^\delta(s)\Psi_n^*(-\mathrm{D}\mathcal{E}(\mathsf{t}(s), \mathsf{u}(s))) \\ & + \frac{\dot{\mathsf{t}}_n(s)}{\lambda_n}\Psi_n\left(\frac{\dot{\mathsf{u}}(s)}{\dot{\mathsf{t}}_n(s)}\lambda_n\right) - \tau_n^\delta(s)\Psi_n\left(\frac{\dot{\mathsf{u}}(s)}{\tau_n^\delta(s)}\right) \end{aligned} \tag{6.25}$$

and prove the following three claims for the terms on the right-hand side.
Claim 6.a.1: there holds

$$\limsup_{n\to\infty} \mathsf{b}^\delta_{\Psi_n}(\tau_n^\delta(s), \dot{\mathsf{u}}(s), -\mathrm{D}\mathcal{E}(\mathsf{t}(s), \mathsf{u}(s))) - \tau_n^\delta(s)\delta \le \mathsf{p}(\dot{\mathsf{t}}(s), \dot{\mathsf{u}}(s), -\mathrm{D}\mathcal{E}(\mathsf{t}(s), \mathsf{u}(s))) \qquad \text{for a.a. } s \in (0,S), \tag{6.26}$$

with p *given by* (5.19). Indeed, the representation formula (5.21) for $\min_{\tau>0} \mathsf{b}^{\delta}_{\Psi_n}$ and estimate (5.26) (cf. Claim 3.3 in the proof of Prop. 5.7) yield

$$\begin{aligned} \mathsf{b}^{\delta}_{\Psi_n}(\tau^{\delta}_n(s), \dot{\mathsf{u}}(s), -\mathrm{D}\mathcal{E}(\mathsf{t}(s), \mathsf{u}(s))) &= \sup\left\{\langle \xi, \dot{\mathsf{u}}(s)\rangle \mid \xi \in K^*_{n,\delta}(-\mathrm{D}\mathcal{E}(\mathsf{t}(s), \mathsf{u}(s)))\right\} \\ &\leq \sup\left\{\|\dot{\mathsf{u}}(s)\|_1 \|\xi\|_\infty \mid \xi \in K^*_{n,\delta}(-\mathrm{D}\mathcal{E}(\mathsf{t}(s), \mathsf{u}(s)))\right\} \\ &\leq \|\dot{\mathsf{u}}(s)\|_1 \left(A \vee \|\mathrm{D}\mathcal{E}(\mathsf{t}(s), \mathsf{u}(s))\|_\infty\right) + \frac{1}{n}\|\dot{\mathsf{u}}(s)\|_1 \log(2en\delta), \end{aligned} \tag{6.27}$$

and we conclude sending $n \to \infty$. Furthermore, we observe that $\tau^{\delta}_n(s)\delta \to 0$ as $n \to \infty$ thanks to the previously proved (6.24).

Claim 6.a.2: *there holds*

$$\limsup_{n\to\infty} \dot{\mathsf{t}}_n(s)\Psi^*_n(-\mathrm{D}\mathcal{E}(\mathsf{t}_n(s)\lambda_n^{-1}, \mathsf{u}(s))) - \tau^{\delta}_n(s)\Psi^*_n(-\mathrm{D}\mathcal{E}(\mathsf{t}(s), \mathsf{u}(s))) \leq 0 \qquad \text{for a.a. } s \in (0, S). \tag{6.28}$$

Indeed, from the uniform Lipschitz continuity of $\mathrm{D}\mathcal{E}(\cdot, u)$ (cf. (5.33)), we gather that

$$\begin{aligned} |\mathrm{D}_i\mathcal{E}(\mathsf{t}_n(s)\lambda_n^{-1}, \mathsf{u}(s))| - |\mathrm{D}_i\mathcal{E}(\mathsf{t}(s), \mathsf{u}(s))| &\leq \left|\mathrm{D}_i\mathcal{E}(\mathsf{t}_n(s)\lambda_n^{-1}, \mathsf{u}(s)) - \mathrm{D}_i\mathcal{E}(\mathsf{t}(s), \mathsf{u}(s))\right| \\ &\leq L_{\mathsf{E}}|\mathsf{t}_n(s)\lambda_n^{-1} - \mathsf{t}(s)| \end{aligned} \tag{6.29}$$

for all $i = 1, \ldots, d$. We now observe that

$$\begin{aligned} |\mathsf{t}_n(s)\lambda_n^{-1} - \mathsf{t}(s)| &\leq |\mathsf{t}_n(s)\lambda_n^{-1} - \mathsf{t}(s)\lambda_n^{-1} + \mathsf{t}(s)\lambda_n^{-1} - \mathsf{t}(s)| \\ &\leq \mathsf{t}(s)(1 - \lambda_n^{-1}) + \lambda_n^{-1}|\mathsf{t}_n(s) - \mathsf{t}(s)| \\ &= T\left(1 - \frac{1}{\lambda_n}\right) + \lambda_n^{-1}\int_0^s \left(\dot{\mathsf{t}}(r) \vee \tau^{\delta}_n(r) - \dot{\mathsf{t}}(r)\right) \mathrm{d}r \\ &\leq T\left(1 - \frac{1}{\lambda_n}\right) + \int_0^s \tau^{\delta}_n(r)\,\mathrm{d}r \end{aligned} \tag{6.30}$$

where we have used the fact that $\lambda_n \geq 1$, the definition of t_n (6.17), and the bound on $\sup_{s\in[0,S]} \dot{\mathsf{t}}(s)$, since t is Lipschitz continuous. We also have

$$\begin{aligned} T(1 - \lambda_n^{-1}) = \frac{T}{T_n}(T_n - T) &\leq \left(\int_0^{\mathsf{s}(0^+)} \tau^{\delta}_n(r)\,\mathrm{d}r + \int_{\mathsf{s}(0^+)}^{\mathsf{s}(T^-)} \left(\dot{\mathsf{t}}(r) \vee \tau^{\delta}_n(r) - \dot{\mathsf{t}}(r)\right) dr + \int_{\mathsf{s}(T^-)}^{S} \tau^{\delta}_n(r)\,\mathrm{d}r\right) \\ &\leq \left(\int_0^S \tau^{\delta}_n(r)\,\mathrm{d}r\right) \leq \sup_{s\in[0,S]} \tau^{\delta}_n(s) \end{aligned} \tag{6.31}$$

again using the definition (6.17) of t_n. Hence, combining estimate (6.29) with (6.30) and (6.31), we gather that

$$\begin{aligned} &|\mathrm{D}_i\mathcal{E}(\mathsf{t}_n(s)\lambda_n^{-1}, \mathsf{u}(s))| - |\mathrm{D}_i\mathcal{E}(\mathsf{t}(s), \mathsf{u}(s))| \leq C \sup_{s\in[0,S]} \tau^{\delta}_n(s) \doteq \bar{C}(n) \quad \text{for all } s \in [0, S], \quad \text{with} \\ &\sup_{n\in\mathbb{N}} n\bar{C}(n) \doteq \overline{C} < \infty, \end{aligned} \tag{6.32}$$

the latter estimate due to (6.24). Therefore, using now the explicit formula (2.6) for Ψ_n^* we get for almost all $s \in (0,S)$ that

$$
\begin{aligned}
&\dot{\mathsf{t}}_n(s)\Psi_n^*(-\mathrm{D}\mathcal{E}(\mathsf{t}_n(s)\lambda_n^{-1},\mathsf{u}(s))) - \tau_n^\delta(s)\Psi_n^*(-\mathrm{D}\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s)))\\
&\overset{(1)}{\leq} \frac{\dot{\mathsf{t}}_n(s)}{n} e^{-nA}\sum_{i=1}^d \cosh(n|\mathrm{D}_i\mathcal{E}(\mathsf{t}_n(s)\lambda_n^{-1},\mathsf{u}(s))|)\\
&\overset{(2)}{\leq} \frac{\dot{\mathsf{t}}_n(s)}{n} e^{-nA}\sum_{i=1}^d \cosh(n|\mathrm{D}_i\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s))| + n\overline{C}(n))\\
&\overset{(3)}{\leq} \frac{d\dot{\mathsf{t}}_n(s)e^{-nA}}{2n} + \frac{\dot{\mathsf{t}}_n(s)}{2n} e^{\overline{C}}e^{-nA}\sum_{i=1}^d e^{n|\mathrm{D}_i\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s))|}\\
&\overset{(4)}{\leq} \begin{cases} \dfrac{d}{n}\dot{\mathsf{t}}_n(s)\left(1+e^{\overline{C}}\right) \doteq \dfrac{C}{n} & \text{for } s \in [\mathsf{s}(0^+),\mathsf{s}(T^-)],\\ C\left(\dfrac{1}{n} + \sup_{s\in[0,S]} \tau_n^\delta(s)\Psi_n^*(-\mathrm{D}\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s)))\right) & \text{for } s \in [0,\mathsf{s}(0^+)) \cup (\mathsf{s}(T^-),S], \end{cases}
\end{aligned}
\tag{6.33}
$$

where (1) follows from the positivity of Ψ_n^* and from the trivial inequality $\cosh(nx) - 1 \leq \cosh(n|x|)$, (2) from (6.29), (3) from (6.32), using that $\cosh(x) \leq \frac{e^x+1}{2}$ for all $x \geq 0$, and (4) is due to the fact that $\|\mathrm{D}\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s))\|_\infty \leq A$ for $s \in [\mathsf{s}(0^+),\mathsf{s}(T^-)]$, and to an elementary inequality on $[0,\mathsf{s}(0^+)) \cup (\mathsf{s}(T^-),S]$. Clearly, $\frac{C}{n} \to 0$; on the other hand, it follows again from (6.24) and the Lipschitz continuity of u that

$$
\sup_{s\in[0,S]} \tau_n^\delta(s)\Psi_n^*(-\mathrm{D}\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s))) \leq \sup_{s\in[0,S]} \frac{d\|\dot{u}(s)\|_\infty \Psi_n^*(-\mathrm{D}\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s)))}{n\left(\Psi_n^*(-\mathrm{D}\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s)))\right)} \leq \frac{C}{n} \to 0 \quad \text{as } n\to\infty. \tag{6.34}
$$

Therefore, (6.28) ensues.

Claim 6.a.3: there holds

$$
\limsup_{n\to\infty} \frac{\dot{\mathsf{t}}_n(s)}{\lambda_n}\Psi_n\left(\frac{\dot{\mathsf{u}}(s)}{\dot{\mathsf{t}}_n(s)}\lambda_n\right) - \tau_n^\delta(s)\Psi_n\left(\frac{\dot{\mathsf{u}}(s)}{\tau_n^\delta(s)}\right) \leq 0 \qquad \text{for a.a. } s \in (0,S). \tag{6.35}
$$

We use the explicit formula (5.17) for Ψ_n, obtaining

$$
\begin{aligned}
&\frac{\dot{\mathsf{t}}_n(s)}{\lambda_n}\Psi_n\left(\frac{\dot{\mathsf{u}}(s)}{\dot{\mathsf{t}}_n(s)}\lambda_n\right)\\
&\overset{\dot{\mathsf{t}}_n\geq\tau_n^\delta}{\leq} \frac{\tau_n^\delta(s)}{\lambda_n}\Psi_n\left(\frac{\dot{\mathsf{u}}(s)}{\tau_n^\delta(s)}\lambda_n\right)\\
&\leq \frac{d\tau_n^\delta(s)\,e^{-nA}}{n} + \sum_{i=1}^d\left[\frac{\dot{\mathsf{u}}_i(s)}{n}\log\left(\lambda_n\frac{\frac{\dot{\mathsf{u}}_i(s)}{\tau_n^\delta(s)} + \sqrt{\left(\frac{\dot{\mathsf{u}}_i(s)}{\tau_n^\delta(s)}\right)^2 + \frac{e^{-2nA}}{\lambda_n^2}}}{e^{-nA}}\right) - \frac{1}{n}\sqrt{\dot{\mathsf{u}}_i(s)^2 + \left(\frac{\tau_n^\delta(s)\,e^{-nA}}{\lambda_n}\right)^2}\right]\\
&\overset{\lambda_n\geq1}{\leq} \tau_n^\delta(s)\Psi_n\left(\frac{\dot{\mathsf{u}}(s)}{\tau_n^\delta(s)}\right) + \sum_{i=1}^d\left[\frac{\dot{\mathsf{u}}_i(s)}{n}\log(\lambda_n) - \frac{1}{n}\sqrt{\dot{\mathsf{u}}_i(s)^2 + \left(\frac{\tau_n^\delta(s)\,e^{-nA}}{\lambda_n}\right)^2} + \frac{1}{n}\sqrt{\dot{\mathsf{u}}_i(s)^2 + (\tau_n^\delta(s)\,e^{-nA})^2}\right],
\end{aligned}
$$

for almost all $s \in (0,S)$, whence

$$
\begin{aligned}
&\dot{\mathsf{t}}_n(s)\frac{1}{\lambda_n}\Psi_n\left(\frac{\dot{\mathsf{u}}(s)}{\dot{\mathsf{t}}_n(s)}\lambda_n\right) - \tau_n^\delta(s)\Psi_n\left(\frac{\dot{\mathsf{u}}(s)}{\tau_n^\delta(s)}\right)\\
&\leq \sum_{i=1}^d \frac{\dot{\mathsf{u}}_i(s)}{n}\log(\lambda_n) - \frac{1}{n}\sqrt{(\dot{\mathsf{u}}_i(s))^2 + \left(\frac{\tau_n^\delta(s)\,e^{-nA}}{\lambda_n}\right)^2} + \frac{1}{n}\sqrt{(\dot{\mathsf{u}}_i(s))^2 + (\tau_n^\delta(s)\,e^{-nA})^2}\,.
\end{aligned}
\tag{6.36}
$$

Observe that the right-hand side of (6.36) tends to zero as $n\to\infty$ taking into account that $\sup_{s\in[0,S]}\|\dot{\mathsf{u}}(s)\|_\infty \leq C$, that $\lambda_n \to 1$, and that $\sup_{s\in[0,S]}\tau_n^\delta(s) \to 0$ by (6.24). This yields (6.35) and, ultimately, (6.21).

Finally, we conclude the integrated lim sup-estimate (6.23) by observing that the Fatou Lemma (cf. (6.22)) applies: this can be checked combining (6.24), (6.25), (6.27) (taking into account that $\sup_{s\in[0,S]}\|\dot{\mathsf{u}}(s)\|_1 \leq C$), (6.33), (6.34), and (6.36).

Step 6b: proof of (6.21) *and* (6.23) *for* Ψ_n *given by* (5.32) *(vanishing-viscosity approximation).* To simplify the notation, in what follows we shall focus on the particular case

$$\varepsilon_n = \frac{1}{n}\,.$$

Preliminarily, we recall that, in the case (5.32),

$$\Psi_n^*(\xi) = \frac{1}{2\varepsilon_n}(\min_{\zeta\in K^*}\|\xi-\zeta\|_*)^2 = \frac{n}{2}d_*(\xi,K^*)^2, \tag{6.37}$$

where $\|\cdot\|_*$ is the dual norm to $\|\cdot\|$, and $d_*(\cdot,K^*)$ denotes the induced distance from the set K^*. Taking into account (6.37), we provide a bound for τ_n^δ from (6.16) again resorting to the Euler-Lagrange equation (5.28). In the present case, it gives

$$(\tau_n^\delta(s))^2 = \frac{\|\dot{\mathsf{u}}(s)\|^2}{2n\delta + n^2 d_*(-\mathrm{D}\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s)),K^*)^2} \qquad \text{for a.a. } s\in(0,S)\,. \tag{6.38}$$

In what follows we will take

$$\delta=\delta_n \quad \text{such that} \quad \delta_n\to\infty \ \text{ and } \ \delta_n\frac{1}{n}\to 0 \ \text{ as } n\to\infty, \tag{6.39}$$

but we will continue to write δ in place of δ_n for shorter notation.

In order to show (6.21), we start from the very same algebraic manipulation as in (6.25) and prove that the terms on the right-hand side converge to the desired limit. We observe that

$$\begin{aligned}
\mathsf{b}_{\Psi_n}^\delta(\tau_n^\delta(s),\dot{\mathsf{u}}(s),-\mathrm{D}\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s))) &= \tau_n^\delta(s)\Psi_n\left(\frac{\dot{\mathsf{u}}(s)}{\tau_n^\delta(s)}\right) + \tau_n^\delta(s)\Psi_n^*(-\mathrm{D}\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s))) + \tau_n^\delta(s)\delta \\
&\overset{(6.38)}{=} \Psi_0(\dot{\mathsf{u}}(s)) + \frac{\|\dot{\mathsf{u}}(s)\|}{2n}\sqrt{2n\delta + n^2 d_*(-\mathrm{D}\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s)),K^*)^2} \\
&\qquad + \frac{n\|\dot{\mathsf{u}}(s)\|d_*(-\mathrm{D}\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s)),K^*)^2}{2\sqrt{2n\delta + n^2 d_*(-\mathrm{D}\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s)),K^*)^2}} + \tau_n^\delta(s)\delta \\
&\overset{n\to\infty}{\to} \Psi_0(\dot{\mathsf{u}}(s)) + \|\dot{\mathsf{u}}(s)\|d_*(-\mathrm{D}\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s)),K^*) \\
&\leq \mathsf{p}(\dot{\mathsf{t}}(s),\dot{\mathsf{u}}(s),-\mathrm{D}\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s)))
\end{aligned}$$

where the last inequality follows from the fact that $\mathrm{D}\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s))\in K^*$ when $\dot{\mathsf{t}}(s)>0$. Thus we conclude the analogue of (6.26). Moreover, observe that, as a consequence of the scaling for δ_n from (6.39), we have

$$\delta \sup_{s\in[0,S]}\tau_n^\delta(s)\to 0 \quad \text{as } n\to\infty. \tag{6.40}$$

We then proceed to show the counterpart to (6.28). The very same calculations as in (6.29) (cf. also (6.31)), give for every $s\in[0,S]$

$$\left|\mathrm{D}_i\mathcal{E}(\mathsf{t}_n(s)\lambda_n^{-1},\mathsf{u}(s)) - \mathrm{D}_i\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s))\right| \leq C\sup_{s\in[0,S]}\tau_n^\delta(s) \leq \frac{C}{\sqrt{n\delta}}\,. \tag{6.41}$$

Resorting now to the explicit formula (6.37) for Ψ_n^* (and using $\xi_n(s)$ and $\xi(s)$ as place-holders for $-\mathrm{D}\mathcal{E}(\mathsf{t}_n(s)\lambda_n^{-1},\mathsf{u}(s))$ and $-\mathrm{D}\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s))$, respectively, to avoid overburdening notation), we get

$$\begin{aligned}
\dot{\mathsf{t}}_n(s)\Psi_n^*(\xi_n(s)) - \tau_n^\delta(s)\Psi_n^*(\xi(s)) &\overset{(1)}{=} \dot{\mathsf{t}}_n(s)\frac{n}{2}d_*(\xi_n(s),K^*)^2 \\
&\overset{(2)}{\leq} \dot{\mathsf{t}}_n(s)\frac{n}{2}\|\xi_n(s)-\xi(s)\|_*^2 \\
&\overset{(3)}{\leq} \frac{C}{\delta} \qquad \text{for a.a. } s\in(\mathsf{s}(0^+),\mathsf{s}(T^-)).
\end{aligned} \tag{6.42}$$

In (6.42), (1) and (2) are due to the fact that $\mathrm{D}\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s)) \in K^*$ for almost all $s \in (\mathsf{s}(0^+),\mathsf{s}(T^-))$, so that $\Psi_n^*(\xi(s)) = 0$, and (3) to estimate (6.41). To prove the analogue of (6.28), we will first treat the case in which $s \in [0,\mathsf{s}(0^+)) \cup (\mathsf{s}(T^-),S]$ (where $\dot{\mathsf{t}}_n(s) = \tau_n^\delta(s)$). Here, we use the Lipschitz estimate (6.41) and the explicit formula for Ψ_n^*. Thus, we find

$$
\begin{aligned}
&\tau_n^\delta(s)\left(\Psi_n^*(\xi_n(s)) - \Psi_n^*(\xi(s))\right)\\
&= \frac{n}{2}\tau_n^\delta(s)\left(d_*(\xi_n(s),K^*)^2 - d_*(\xi(s),K^*)^2\right)\\
&\leq \frac{n}{2}\tau_n^\delta(s)\left(\left(d_*(\xi_n(s),\xi(s)) + d_*(\xi(s),K^*)\right)^2 - d_*(\xi(s),K^*)^2\right)\\
&\leq \frac{n}{2}\tau_n^\delta(s)\left(d_*(\xi_n(s),\xi(s))^2 + 2d_*(\xi(s),K^*)d_*(\xi_n(s),\xi(s))\right)\\
&\overset{(6.41)}{\leq} n\tau_n^\delta(s)\left(\frac{C}{n\delta} + \frac{C}{\sqrt{n\delta}}\right) \qquad \text{for all } s \in [0,\mathsf{s}(0^+)) \cup (\mathsf{s}(T^-),S]
\end{aligned}
\tag{6.43}
$$

Combining (6.42) and (6.43) we infer (6.28), since $\delta = \delta_n \to \infty$ and $\frac{\delta_n}{n} \to 0$ as $n \to \infty$. In order to prove the analogue of (6.35), we use the explicit formula (5.32) of Ψ_n obtaining

$$
\begin{aligned}
\frac{\dot{\mathsf{t}}_n(s)}{\lambda_n}\Psi_n\left(\frac{\dot{\mathsf{u}}(s)}{\dot{\mathsf{t}}_n(s)}\lambda_n\right) &\overset{\dot{\mathsf{t}}_n \geq \tau_n^\delta}{\leq} \frac{\tau_n^\delta(s)}{\lambda_n}\Psi_n\left(\frac{\dot{\mathsf{u}}(s)}{\tau_n^\delta(s)}\lambda_n\right) = \Psi_0(\dot{\mathsf{u}}(s)) + \frac{\lambda_n}{2n\tau_n^\delta(s)}\|\dot{\mathsf{u}}(s)\|^2\\
&= \tau_n^\delta(s)\Psi_n\left(\frac{\dot{\mathsf{u}}(s)}{\tau_n^\delta(s)}\right) + (\lambda_n - 1)\frac{\|\dot{\mathsf{u}}(s)\|^2}{2n\tau_n^\delta(s)}.
\end{aligned}
$$

It follows from (6.38) and (6.39) that, for n sufficiently big,

$$
n\tau_n^\delta(s) \geq \frac{\|\dot{\mathsf{u}}(s)\|}{2 + d_*(-\mathrm{D}\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s)),K^*)},
$$

hence we deduce that

$$
\frac{\dot{\mathsf{t}}_n(s)}{\lambda_n}\Psi_n\left(\frac{\dot{\mathsf{u}}(s)}{\dot{\mathsf{t}}_n(s)}\lambda_n\right) - \tau_n^\delta(s)\Psi_n\left(\frac{\dot{\mathsf{u}}(s)}{\tau_n^\delta(s)}\right) \leq (\lambda_n - 1)C \qquad \text{for a.a. } s \in (0,S). \tag{6.44}
$$

Then, (6.35), and ultimately (6.21), ensue, since $\lambda_n \to 1$.

It remains to verify the integrated inequality (6.23). Taking into account (6.25), we observe that the *uniform* (w.r.t. $s \in (0,S)$) estimates (6.40), (6.42), (6.43), and (6.44), immediately allow for the application of the Fatou Lemma (6.22). Finally, we observe that, again by the general representation formula (5.21), there holds

$$
\begin{aligned}
\mathsf{b}_{\Psi_n}(\tau_n^\delta(s),\dot{\mathsf{u}}(s),-\mathrm{D}\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s))) &= \sup\left\{\langle\xi,\dot{\mathsf{u}}(s)\rangle \mid \xi \in K^*_{n,\delta}(-\mathrm{D}\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s)))\right\}\\
&\leq \sup\left\{\|\dot{\mathsf{u}}(s)\|_1\|\xi\|_\infty \mid \xi \in K^*_{n,\delta}(-\mathrm{D}\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s)))\right\}\\
&\leq C\|\dot{\mathsf{u}}(s)\|_1\left(\|\mathrm{D}\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s))\|_\infty + \sqrt{\frac{\delta}{n}}\right),
\end{aligned}
$$

and we conclude the integral estimate from the boundedness of $\|\dot{\mathsf{u}}(s)\|_1$, $\|\mathrm{D}\mathcal{E}(\mathsf{t}(s),\mathsf{u}(s))\|_\infty$ and sending $n \to \infty$. Thus, (6.23) is proven.

Step 7: recovery sequence for a general curve u and conclusion of the proof. Now we construct a recovery sequence for a curve with countably many jumps, following the argument from the proof of [BP16, Thm. 4.2]. Given the jump set J_u, fix $\varepsilon > 0$, consider a countable set $\{t^i\}_{i\in I} \subseteq \mathrm{J}_u \cup \{0,T\}$ (with $t^i < t^{i+1}$ for all $i \in I$) such that

$$
\mathrm{Jmp}_{\mathsf{p},\mathcal{E}}(u;[0,T] \setminus \{t^i\}) < \varepsilon, \tag{6.45}
$$

and such that the interval $[0,T]$ can be written as the union of disjoint subintervals

$$
[0,T] = \bigcup_{i\in I}\Sigma^i \qquad \text{where } \Sigma^i = [t^i,t^{i+1}].
$$

Then, we let $t_n^i = \mathsf{t}_n(\mathsf{s}(t^i))$, and set

$$\lambda_n^i := \frac{t_n^{i+1} - t_n^i}{t^{i+1} - t^i}.$$

We define the recovery sequence by

$$u_n(t) := \mathsf{u}\left(\mathsf{t}_n^{-1}\left(\lambda_n^i(t - t^i) + t_n^i\right)\right) \qquad \text{for } t \in \Sigma^i, \tag{6.46}$$

so that

$$\dot{u}_n(t) = \frac{\dot{\mathsf{u}}}{\dot{\mathsf{t}}_n}\left(\mathsf{t}_n^{-1}\left(\lambda_n^i(t - t^i) + t_n^i\right)\right)\lambda_n^i \qquad \text{for } t \in (\Sigma^i)^\circ.$$

We have now that

$$\begin{aligned}
&\int_0^T \Big[\Psi_n\left(\dot{u}_n(t)\right) + \Psi_n^*\left(-\mathrm{D}\mathcal{E}(t, u_n(t))\right)\Big]\,\mathrm{d}t \\
&\quad = \sum_i \int_{\Sigma^i} \left[\Psi_n\left(\frac{\dot{\mathsf{u}}}{\dot{\mathsf{t}}_n}\left(\mathsf{t}_n^{-1}\left(\lambda_n^i(t - t^i) + t_n^i\right)\right)\lambda_n^i\right) + \Psi_n^*\left(-\mathrm{D}\mathcal{E}\big(t, \mathsf{u}(\mathsf{t}_n^{-1}(\lambda_n^i(t - t^i) + t_n^i))\big)\right)\right]\mathrm{d}t \\
&\quad = \sum_i \int_{\mathsf{s}(t^i)}^{\mathsf{s}(t^{i+1})} \left[\Psi_n\left(\frac{\dot{\mathsf{u}}}{\dot{\mathsf{t}}_n}(s)\lambda_n^i\right) + \Psi_n^*\Big(-\mathrm{D}\mathcal{E}((\lambda_n^i)^{-1}(\mathsf{t}_n(s) - t_n^i) + t^i, \mathsf{u}(s))\Big)\right]\frac{\dot{\mathsf{t}}_n(s)}{\lambda_n^i}\,\mathrm{d}s.
\end{aligned}$$

Applying estimate (6.21) in every subinterval $[\mathsf{s}(t^i), \mathsf{s}(t^{i+1})]$ and Fatou's Lemma (cf. (6.22)) on the whole interval $[0, S]$, we obtain inequality (6.15).

The convergence of the variations again follows by the definition of u_n.

The pointwise convergence $u_n(t) \to u(t)$ for $t \in [0,T] \setminus \mathrm{J}_u$ is again trivial. The following calculations show that, by construction, the convergence holds also in the points $\{t^i\} \subseteq \mathrm{J}_u$. Indeed,

$$u_n(t^i) \overset{(6.46)}{=} \mathsf{u}\left(\mathsf{t}_n^{-1}\left(t_n^i\right)\right) = \mathsf{u}\left(\mathsf{t}_n^{-1}\left(\mathsf{t}_n(\mathsf{s}(t^i))\right)\right) = \mathsf{u}\left(\mathsf{s}(t^i)\right) \overset{(6.12)}{=} u(t^i),$$

while from (6.45) and the convergence of the variations we have that

$$\lim_{n\to\infty} |u_n(t) - u(t)| < \varepsilon \qquad \text{for all } t \in \mathrm{J}_u \setminus \{t^i\}.$$

In fact, the recovery sequence u_n has a hidden dependence on ε. Then taking $\varepsilon = n^{-1}$ we define a new recovery sequence, that we keep labelling u_n, and sending $n \to \infty$ (ε to zero) we conclude.

This finishes the proof of Theorem 5.8. ∎

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